Finite Supermodular Design with Interdependent Valuations

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Abstract

This paper studies supermodular mechanism design in environments with arbitrary (finite) type spaces and interdependent valuations. In these environments, the designer may have to use Bayesian equilibrium as a solution concept, because ex post implementation may not be possible. We propose direct (Bayesian) mechanisms that are robust to certain forms of bounded rationality while controlling for equilibrium multiplicity. In quasi-linear environments with informational and allocative externalities, we show that any Bayesian mechanism that implements a social choice function can be converted into a supermodular mechanism that also implements the original decision rule. The proposed supermodular mechanism can be chosen in a way that minimizes the size of the equilibrium set, and we provide two sets of sufficient conditions to this effect: for general decision rules and for decision rules that satisfy a certain requirement. This is followed by conditions for supermodular implementation in unique equilibrium.

Keywords: Implementation, supermodular mechanisms, multiple equilibrium problem, learning, strategic complementarities, supermodular games.

JEL Classification: C72, D78, D83.

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1 Introduction

This paper studies supermodular mechanism design in environments with interdependent valuations and arbitrary (in particular, multidimensional) finite type spaces. This approach was introduced by Mathevet (25) in differentiable quasilinear environments with private values and one-dimensional types.[1] The main motivation is to design direct mechanisms that are robust to certain forms of bounded rationality while controlling for equilibrium multiplicity. It is important to extend supermodular mechanism design to environments with informational and allocative externalities and multidimensional types for at least two reasons. First, these environments capture many realistic situations. Second, it is often impossible to use dominant strategy or ex post implementation in these settings (see Jehiel *et al.* (22) and Section 2), and thus the designer may resort to Bayesian equilibrium as a solution concept. It becomes useful to have a simple method for improving the behavioral robustness of Bayesian mechanisms.

In this paper, we are concerned with the design of supermodular mechanisms whose equilibrium set is of minimal size. We call this minimal supermodular implementation. Supermodular mechanism design aims to induce the right incentives so that agents play a supermodular game. Supermodular games are games where players have monotone best responses, i.e. each player wants to play a "larger" strategy if others do so as well. On the theoretical front, the reasons for using supermodular mechanisms stem from Milgrom and Roberts (27), (28) and Vives (33): supermodular games have extremal equilibria, a smallest and a largest one, that enclose all the iteratively undominated strategies and all the limit points of all adaptive and sophisticated learning dynamics. Therefore, supermodular games are robust to a wide range of behaviors, including boundedly rational behaviors. In particular, if the designer had the opportunity to use her mechanism repeatedly, then adaptive learners (Milgrom and Roberts (27)) would end up within the interval prediction, which is the interval between the extremal equilibria. Therefore, the objective of minimizing the size of the interval prediction has several virtues. It minimizes the multiple equilibrium problem, since all equilibria are contained in it.[2] It also guarantees a more accurate convergence of the learning dynamics. Ideally, this interval reduces to a single point in certain situations (see Section 2), thereby solving the multiplicity issue and ensuring convergence of all dynamics.

Supermodular mechanisms have other attractive theoretical properties. Not only are their mixed strategy equilibria unstable (Echenique and Edlin (17)), which justifies

[1] Chen (8) was the first to propose a supermodular mechanism (to implement the Lindahl correspondence).

[2] If the outcome function of the mechanism is continuous and if the interval prediction is tight, then all equilibrium outcomes are close, so that the output of the mechanism must be close to the socially desirable objective.

ruling them out of the analysis, but many pure equilibria are stable, such as the extremal equilibria (Echenique (16)). Thus, a perturbation should not destabilize permanently a socially desirable alternative implemented via a supermodular mechanism (provided the underlying equilibrium was stable).

The robustness properties of supermodular mechanisms have been corroborated by several experiments. Chen and Gazzale (11) run experiments on a game for which they control the amount of supermodularity. They show how convergence in that game is significantly better when it is supermodular. Healy (18) tests five public goods mechanisms and he observes that subjects learn to play the equilibrium only in those mechanisms that induce a supermodular game. Other experiments (e.g. Chen and Plott (9), and Chen and Tang (10)) provide results emphasizing the importance of dynamic convergence in the context of implementation. Most of these experiments demonstrate that convergence to an equilibrium is not a trivial issue.

In this paper, we generalize supermodular mechanism design to environments with allocative externalities, interdependent valuations (i.e. informational externalities) and arbitrary (finite) type spaces. There are two important reasons for doing so.

Firstly, it allows covering mechanism design problems of interest. The importance of allocative externalities is well documented in the literature. Jehiel and Moldovanu (20) use patent licensing in an oligopolistic market as an example. Informational externalities are also a realistic assumption, proved to be interestingly challenging by many papers (Cremer and McLean (13), Maskin (24), Dasgupta and Maskin (14), Perry and Reny (30), Chung and Ely (12), Bergemann and Morris (2), etc). Finally, it is often natural to interpret information as a multidimensional type in many design problems. Consider, for example, oil companies bidding to obtain a drilling permit. Their private information is modeled as a multidimensional signal (e.g. expected amount of oil in the oil field, proximity to other reserves, etc).

Secondly, it is difficult to use dominant strategy or ex post implementation in those environments — with allocative externalities, interdependent valuations and multidimensional types — and thus behaviorally-robust Bayesian mechanisms become especially appealing. In quasilinear environments with interdependent valuations and multidimensional types, many impossibility results limit the set of available solution concepts. The conclusions are rather pessimistic about dominant strategy equilibrium and ex post equilibrium. Williams and Radner (34) show that efficient dominant strategy implementation is generally not possible when agents have interdependent valuations. Jehiel *et al.* (22) prove a strong impossibility result: when types are multidimensional and valuations are interdependent, only trivial decision rules can generically be implemented in ex post equilibrium. If the designer wants to implement a meaningful social choice function, not even necessarily efficient, she may have to use Bayesian equilibrium as a solution concept (see Section 2). Even then, impossibility results exist. Jehiel and Moldovanu (21) show that it is difficult to reconcile Bayesian incentive compatibility with some efficiency constraint. These negative results indicate that Bayesian equilibrium may often be a natural candidate as a solution concept. However, playing a Bayesian equilibrium requires more, in general, on the part of the agents. Agents have to be Bayesian rational, and the information structure and rationality have to be common knowledge among the agents (Brandenburger and Dekel (5)). As other Nash-related concepts, Bayesian equilibrium calls for correct predictions of opponents' play to determine one's own strategy. In this context, the ability to construct supermodular Bayesian mechanisms is attractive, because eventual play of some equilibrium can be achieved by unsophisticated agents who follow simple behavioral rules.

Our paper provides methods for converting *any* truthful Bayesian mechanism into a (truthful) supermodular mechanism whose equilibrium set is of minimal size. The idea is to create complementarities between agents' announcements by augmenting the original transfer scheme with a function. This function vanishes in expectation and therefore preserves incentive compatibility. Although there exist many ways in which a mechanism can be transformed into a supermodular mechanism, we derive the one that most adequately addresses the multiple equilibrium problem. To this purpose, we add just enough strategic complementarities to ensure that a supermodular game is induced, but not in any excess of that level.

We present two sets of results for minimal supermodular implementation. In both instances, "best" is used to designate the mechanism with the smallest interval prediction. The first result shows that if a social choice function is implementable, then its decision rule can be implemented by the best supermodular mechanism among all the supermodular mechanisms *whose transfers are in a certain class*. No additional condition is required. In particular, this result holds for all (implementable) decision rules and all valuation functions. The result also provides an explicit transfer scheme. The second result characterizes the *overall* best supermodular mechanism among all possible supermodular mechanisms or transfers: if a social choice function is implementable and satisfies a reducibility condition, then its decision rule can be implemented by the (overall) best supermodular mechanism. Although the first result reaches a weaker conclusion than the second, it applies under very general conditions. Finally, we provide conditions under which truthtelling is the essentially unique equilibrium. For fine (or rich) type spaces, this ensures excellent stability properties: all learning dynamics converge to the truthful equilibrium, and the game is dominance solvable.

Beyond the generalizations of supermodular mechanism design, this paper provides new insights into the design of minimally supermodular mechanisms. The use of finite types clarifies the existence and the construction of these mechanisms. For example, we show that the problem of building minimally supermodular mechanisms is equivalent to solving a system of linear equations. This allows the application of numerical methods for designing these mechanisms. Further, it becomes possible to derive necessary and sufficient conditions for the existence of minimal transfers. Nonetheless, these conditions are not practical, and they do not always come with explicit formulas for the transfers. To remedy these shortcomings, we propose a simple sufficient condition under which transfers take a closed form.

A number of papers are related to our work. The first paper to present a supermodular mechanism was Chen (8). Mathevet (25) developed supermodular mechanism design as a general method under incomplete information. Since his paper is the closest to ours, our contribution deserves clarification. As already said, our environment is more general, due to the interdependent values and the multidimensional types, although this comes at the cost of finite types. In Mathevet (25)'s environment, dominant strategy implementation is still a powerful tool. This paper also clarifies the construction of minimally supermodular mechanisms, especially with our reducibility condition and the formulation of this problem as a linear system. Finally, we propose different options for minimal supermodular design when sufficient conditions fail, while Mathevet (25) does not. In particular, our first main result always applies, and the formulation as a linear system can yet allow finding solutions. Cabrales and Serrano (7) study implementation with boundedly rational agents who follow adaptive better response dynamics. This learning dynamics excludes learning processes that we study here. Finally, our paper is related to the literature on rationalizable implementation (e.g. Abreu and Matsushima (1), and Bergemann, Morris, and Tercieux (4)), because this solution concept has the potential to imply nice learning properties when a unique equilibrium is rationalizable. Abreu and Matsushima (1) show that any social choice function can be virtually implemented in iteratively undominated strategies. Their result is very powerful but their mechanism remains complex, as the dimension of the message space must be arbitrarily large. Experimental evidence does not support this mechanism (Sefton and Yavas (31)). Instead we look at direct mechanisms and exact implementation. In general, the concept of rationalizability is such that a strategy may not be rationalizable because it is dominated by another dominated strategy, an argument a la Jackson (19). For example, in Bergemann, Morris, and Tercieux (4), the best responses are not well-defined off-equilibrium, and off-equilibrium behaviors are one of our motivations.

The remainder of the paper is organized as follows. A motivating example is presented in Section 2. Section 3 defines the framework of supermodular mechanism design. Section 4 introduces the notion of minimal implementation and contains our two main results. Section 5 studies supermodular implementation in unique equilibrium. Concluding remarks appear in Section 6. All proofs are relegated to the Appendix.

2 Motivating Example

This section illustrates our approach in a simple public good example. In this example, the designer would like to implement an efficient allocation, but this cannot be done in ex post equilibrium (hence in dominant strategies). Thus, the designer may decide to work with Bayesian implementation. We show how the designer can start from *any* truthful Bayesian mechanism, in particular one with poor stability properties, and turn it into a (truthful) supermodular mechanism with a unique equilibrium.

Consider a social planner who has to make a decision between two public goods, A or B, in a society that consists of two agents 1 and 2.[3] Each agent i has a type $\theta_i = (\theta^i_A, \theta^i_B)$ in $\{(2,1),(1,2)\}$. Types are drawn with the following probabilities, which is common knowledge: $\Pr(\theta_1 = (2,1)) = 7/20$ and $\Pr(\theta_2 = (2,1)) = 9/10$. Agent i's valuation for public good $g \in \{A, B\}$ at types $\theta = (\theta_1, \theta_2)$ is $V^i_g(\theta)$. The valuations are given in the following matrix (rows represent 1's type or report):

$V(\theta)$	$(2,1)$	$(1, 2)$
$(2,1)$	$V^1_A = 0 \quad V^1_B = .24$ $V^2_A = 0 \quad V^2_B = .01$	$V^1_A = 0 \quad V^1_B = .05$ $V^2_A = 0 \quad V^2_B = -.09$
$(1,2)$	$V^1_A = 0 \quad V^1_B = -.11$ $V^2_A = 0 \quad V^2_B = .1$	$V^1_A = 0 \quad V^1_B = -.08$ $V^2_A = 0 \quad V^2_B = .97$

The valuations for alternative A are always zero. Moreover, each agent's type affects both agents' valuations for alternative B. In particular, agent 1's valuation for good B is larger when her type matches agent 2's type, i.e. when $\theta^1 = \theta^2$. Agent 2's valuation for good B, however, is always larger when 1's type is $(1,2)$, regardless of her true type.

The efficient decision rule is

$x(\hat{\theta})$	$(2,1)$	$(1, 2)$
$(2,1)$	B	A
$(1,2)$	A	B

which we assume the designer would like to implement. Denote agent i's transfers as a function of reported types by:

[3] Throughout the rest of the paper we will refer to the designer (or planner) and agent 1 as "she", and agent 2 as "he".

$t_i(\hat{\theta})$	$(2,1)$	$(1,2)$
$(2,1)$	t_i^1	t_i^2
$(1,2)$	t_i^3	t_i^4

The efficient decision rule is not ex post incentive compatible.[4] To see why, let us consider the ex post incentive compatibility conditions for agent 1 when agent 2's type is $\theta_2 = (1,2)$ and is truthfully reported. At type $\theta_1 = (2,1)$, ex post incentive compatibility for agent 1 requires

$$t_1^2 \geq .05 + t_1^4.$$

At type $\theta_1 = (1,2)$, ex post incentive compatibility requires

$$-.08 + t_1^4 \geq t_1^2.$$

The last two inequalities cannot be jointly satisfied, which proves that the efficient decision rule is not ex post implementable. The designer is therefore inclined to work with Bayesian implementation. We proceed to show that there exist transfers that implement the efficient decision rule in Bayesian equilibrium.

Bayesian incentive compatibility for agent 1 requires that truthtelling be weakly preferred to lying when her true type is (2, 1)

$$.9(.24 + t_1^1) + .1(0 + t_1^2) \geq .9(0 + t_1^3) + .1(.05 + t_1^4)$$

and when her true type is (1,2)

$$.9(0 + t_1^3) + .1(-.08 + t_1^4) \geq .9(-.11 + t_1^1) + .1(0 + t_1^2).$$

Combining these two inequalities, we obtain that for any t_1 such that

$$.1 \geq .9(t_1^1 - t_1^3) + .1(t_1^2 - t_1^4) \geq -.2$$

the efficient decision rule is Bayesian incentive compatible for agent 1. Similarly, Bayesian incentive compatibility is satisfied for any t_2 such that

$$.7 \geq .35(t_2^1 - t_2^2) + .65(t_2^3 - t_2^4) \geq .1.$$

In particular, the designer can choose:

[4] Ex post incentive compatibility requires that for all i and θ, $u_i(x(\theta),\theta) \geq u_i(x(\theta_i', \theta_{-i}), \theta)$ for all θ_i'. This means that if all other agents report truthfully, truthtelling is a best response for each agent i at every possible realizations of types θ.

$t_1(\hat{\theta})$	$(2,1)$	$(1, 2)$
$(2,1)$	-1	10
$(1,2)$	1	-7

$t_2(\hat{\theta})$	$(2,1)$	$(1, 2)$
$(2,1)$	3	-3.5
$(1,2)$	-1	2

As we will see, the magnitude of these transfers is large enough to offset any consideration about the valuations. Given these transfers, the resulting payoff matrix in the ex ante Bayesian game is:

EU	truthtelling	constant (2,1)	constant (1,2)	always lie
truthtelling	.2*; .4*	.4; .4	-1.1; .2	-1; .2
constant (2,1)	.1; 2.4	-1; 3.1*	10*; -3.5	8.9*; -2.8
constant (1,2)	.2;-.6	1*; -1	-7; 2.1*	-6.2; 1.8
always lie	.1; 1.3	-.4; 1.7*	4.1; -1.6	3.7;-1.2

where row entries and first payoffs correspond to agent 1, while column entries and second payoffs correspond to agent 2. Best responses are denoted by asterisks. The game described by this payoff matrix is not (ex ante) dominance solvable. Despite being the unique equilibrium, truthtelling is unstable; after a small perturbation, convergence to it fails under various dynamics. The intuition goes as follows. If agent 2 plays the constant announcement $(1, 2)$ irrespective of his true type, then agent 1 will best-respond by announcing (2, 1) regardless of her type. In return, agent 2 will also announce (2, 1) for every type. Then agent 1 will want to play the constant announcement (1, 2), followed by a constant announcement of (1, 2) by agent 2. We are back to the original strategy of agent 2. These transfers give rise to cycling behaviors and the problem extends beyond best-response dynamics.

To overcome this problem, we propose converting the mechanism into a supermodular mechanism with the smallest equilibrium set. The idea is to modify the original transfers in a way that adds complementarity between agents' announcements, but not so much as to create multiple equilibria. In Section 4, we provide the formula for this transformation. When applied to the current example, the formula outputs

$t_1^{SM}(\hat{\theta})$	$(2,1)$	$(1, 2)$
$(2,1)$	.1	.1
$(1,2)$	.18	.4

$t_2^{SM}(\hat{\theta})$	$(2,1)$	$(1, 2)$
$(2,1)$	.4	-.04
$(1,2)$	.4	.14

which translates into the ex ante payoff matrix:

EU	truthtelling	constant (2,1)	constant (1,2)	always lie
truthtelling	.2*; .4*	.2*; .4	.2; .2	.2; .2
constant (2,1)	.1; .4	.1; .5*	.1; -.04	.1; .1
constant (1,2)	.2; .4*	.2; .4	.4*; .3	.4*;.2
always lie	.1; .4	.1 .5*	.3; .02	.3; .1

This payoff matrix describes a supermodular game — assuming $(1,2) > (2,1)$ — in which truthtelling is the unique equilibrium. Supermodularity means that, for every true type, each agent wants to make larger announcements (under the assumed order) if the other agent does so as well. This mechanism has nice properties. The reader can verify that iterative deletion of strictly dominated strategies gives a unique prediction, truthtelling. By Milgrom and Roberts (27), all adaptive learning dynamics converge to the truthful equilibrium, wherever they are initiated. The original instability problem is solved. In Section 7.1 of the Appendix, we present another version of this example where the designer starts with transfers that create multiple equilibria. In this case, our transformation technique delivers a supermodular mechanism in which truthtelling is the unique equilibrium.

3 Finite Supermodular Design: The Framework

Consider n agents, each endowed with quasilinear preferences over a set of alternatives. The set of players will be denoted by N. An alternative is a vector $(x,t) = (x_1, \dots, x_n, t_1, \dots, t_n)$, where x_i is an element of a set $X_i \subset \mathbb{R}^{m_i}$, x is an element of $X = \prod_{i=1}^{n} X_i$, and $t_i \in \mathbb{R}$ for all $i \in N$. In this environment, x_i is interpreted as agent i's allocation, x is the complete allocation profile, and t_i is the money transfer i receives.

Each agent i has a finite type space Θ_i with generic element θ_i. The types of agents other than i are denoted by $\theta_{-i} \in \Theta_{-i} \equiv \prod_{j \neq i} \Theta_j$, and $\theta \in \Theta \equiv \prod_{i \in N} \Theta_i$ denotes a full type profile. There are no restrictions on the nature of the type spaces: each Θ_i could be, for example, a subset of $\mathbb{R}$, $\mathbb{R}^n$, or any other finite collection of elements. Information is incomplete. There is a common prior with probability mass function ϕ on Θ known to the mechanism designer. Types are assumed to be independently distributed, and ϕ has full support.

A mechanism designer wishes to implement an allocation for each realization of types. This objective is represented by a decision rule $x : \Theta \mapsto (x_i(\theta))_{i=1}^{n}$. To this end, the designer sets up a transfer scheme $t_i : \Theta \to \mathbb{R}$ for each i. A mechanism is denoted by $\Gamma = (\{\Theta_i\}, (x,t))$. Agents are asked to announce a type, and from the vector of announced types, an allocation and a transfer accrue to each agent. The pair $f = (x,t)$ is called a social choice function. We adopt the conventional notation where $\hat{\theta}_i$ is agent i's announced type, $\hat{\theta}_{-i}$ is the vector of announced types of all agents but i, and $\hat{\theta}$ denotes the announced types of all agents.

Each agent i's preferences over alternatives are represented by a utility function $u_i(x, t_i, \theta) = V_i(x;\theta) + t_i$, where $V_i : X \times \Theta \to \mathbb{R}$ is referred to as i's valuation.

This formulation allows for allocational externalities, as V_i can also depend on the allocations of agents other than i. It also captures the case of informational externalities (interdependent valuations) since the valuations may depend on everyone's types. Agent i's utility function at type θ in Γ is $u_i^\Gamma(\hat{\theta};\theta) = V_i(x(\hat{\theta});\theta) + t_i(\hat{\theta})$. A pure strategy for agent i under incomplete information is a function $\hat{\theta}_i : \Theta_i \to \Theta_i$ that maps true types into announced types. Strategy $\hat{\theta}_i(\cdot)$ is called a deception. Agent i's (ex ante) utility function in Γ is $U_i^\Gamma(\hat{\theta}_i(.), \hat{\theta}_{-i}(.)) = E_\theta[u_i^\Gamma(\hat{\theta}(\theta);\theta)]$.

A partial order $\geq$ on a set X is a binary relation that satisfies reflexivity, antisymmetry, and transitivity (see Topkis (32)). The couple $(X, \geq)$ is referred to as a partially ordered set. For $x, y \in X$, if $y \geq x$ and $y \neq x$, then we write $y > x$. A total order on set X is a binary relation that satisfies comparability, antisymmetry, and transitivity.[5] If $\geq$ is a total order on X, then $(X, \geq)$ is called a totally ordered set.

An order $\geq^*$ on set X is a linear extension of a partial order $\geq$ if (i) $(X, \geq^*)$ is a totally ordered set and (ii) for every x, y in X, if $y \geq x$, then $y \geq^* x$. Elements that are ordered under $\geq$ remain identically ordered under $\geq^*$, but $\geq^*$ also orders all the elements that are unordered under $\geq$.

Suppose that $(X, \geq_X)$ and $(Y, \geq_Y)$ are partially ordered sets. A function $h : X \times Y \to \mathbb{R}$ has increasing (decreasing) differences in (x, y) on $X \times Y$ if for all $x'' \geq_X x'$ and all $y'' \geq_Y y'$, $h(x'', y'') - h(x', y'') \geq (\leq) h(x'', y') - h(x', y')$. In game-theoretic models, increasing differences translate the notion of strategic complementarity.

Take x, x' in a partially ordered set $(X, \geq)$. If x and x' have a least upper bound (greatest lower bound) in X, it is referred to as their join (meet) and denoted by $x \vee x'$ $(x \wedge x')$. A lattice is a partially ordered set that contains the join and meet of every pair of its elements. Given a lattice X, a function $h : X \to \mathbb{R}$ is supermodular if $h(x) + h(x') \leq h(x \vee x') + h(x \wedge x')$ for all x and x' in X.

A finite game is a tuple $(N, \{(\Theta_i, \geq_i)\}, \{w_i\})$ where N is a finite set of players, $(\Theta_i, \geq_i)$ is a partially ordered strategy set with finitely many elements for each i, and $w_i : \Theta \to \mathbb{R}$ is Player i's payoff function.

Definition 1 *A finite game $\mathcal{G} = (N, \{(\Theta_i, \geq_i)\}, \{w_i\})$ is supermodular if for all $i \in N$, (1) $(\Theta_i, \geq_i)$ is a lattice, (2) w_i has increasing differences in $(\hat{\theta}_i, \hat{\theta}_{-i})$ on (Θ_i, Θ_{-i}), and (3) w_i is supermodular in $\hat{\theta}_i$ on Θ_i for each $\hat{\theta}_{-i} \in \Theta_{-i}$.*

The rest of the paper focuses on totally ordered sets $(\Theta_i, \geq_i)$. In this case, requirements (1) and (3) in definition are trivially satisfied and we only need to satisfy (2) to ensure that the game is supermodular.

[5] Comparability means that $x \geq y$ or $y \geq x$ for all x, y in X. Note that comparability implies reflexivity; hence, every total order is also a partial order.

The game induced by mechanism Γ can be formulated at three stages: Ex ante, interim, and ex post (complete information). Let us denote the game induced ex post by $\mathcal{G}(\theta) = (N, \{\Theta_i, \geq_i\}, \{u_i^\Gamma(\cdot\,; \theta)\})$. Let $\mathcal{G} = (N, \{\Theta_i^{\Theta_i}, \geq_i\}, \{U_i^\Gamma\})$ be the ex ante Bayesian game induced by Γ. Among these three formulations, the paper considers supermodularity at the ex post level, because this is the strongest requirement. If the ex post game is supermodular for all possible realizations of types θ, then the game will be supermodular in its ex ante and interim formulations.

Definition **2** *A social choice function $f = (x, t)$ is (truthfully) supermodular implementable if truthtelling, i.e. $\hat{\theta}_i(\theta_i) = \theta_i$ for all i, is a Bayesian equilibrium of $\mathcal{G}$ and if $\mathcal{G}(\theta)$ is supermodular for each θ.*

4 Minimal Supermodular Implementation

In this section, we present two results dealing with minimally supermodular mechanisms. The main issue with supermodular implementation lies in finding the appropriate amount of complementarity to add to a mechanism. While complementarities lead to good dominance and learning properties, via the monotonicity of the best responses, excessive complementarities may generate multiple equilibria. Therefore, one negative consequence might be enhancing the "learnability" of undesirable equilibria. In our model, only the truthful equilibrium outcome is known to be socially desirable, hence it may be easier for agents to learn, but they may learn to play an untruthful equilibrium.

This section is organized as follows. First, we present the foundational results and concepts that underlie the method behind minimal supermodular implementation. Then we present our results in separate sections. Our first result is that, for *any* implementable social choice function, its decision rule can be minimally supermodular implemented by transfers within a class. This result holds for *all* valuation functions. This is a strong result, conditionally on choosing transfers within the class. Our second result does not restrict attention to a class of mechanisms or transfers. For *any* valuation functions, if a social choice function is implementable and satisfies a sufficient condition, then its decision rule can be minimally supermodular implemented among *all* transfer functions making the mechanism supermodular. Both results provide explicit expressions for the transfers.

Let us introduce some definitions. For all i, let $(\geq_i^1, \geq_i^2)$ be a pair of orders such that $\geq_i^1$ is a total order on Θ_i and $\geq_i^2$ is a total order on Θ_{-i}. Let $\geq_{-i}$ be the product order on Θ_{-i} obtained from $\{\geq_j^1\}$: $\theta''_{-i} \geq_{-i} \theta'_{-i}$ iff $\theta''_j \geq_j^1 \theta'_j$ for all $j \neq i$. A profile of orders $\{(\geq_i^1, \geq_i^2)\}_i$ is *consistent* if for all i, $\geq_i^2$ is a linear extension of $\geq_{-i}$ on Θ_{-i}.

4.1 Foundations

Mathevet (25) relates the degree of complementarities to the size of the equilibrium set via the following binary relation.

Definition 3 *The binary relation $\succeq_{ID}$ on the space of transfer functions is defined such that $\tilde{t} \succeq_{ID} t$ if for all $i \in N$ and for all $\theta_i'' \geq_i^1 \theta_i'$ and $\theta_{-i}'' \geq_{-i} \theta_{-i}'$, $\tilde{t}_i(\theta_i'', \theta_{-i}'') - \tilde{t}_i(\theta_i'', \theta_{-i}') - \tilde{t}_i(\theta_i', \theta_{-i}'') + \tilde{t}_i(\theta_i', \theta_{-i}') \geq t_i(\theta_i'', \theta_{-i}'') - t_i(\theta_i'', \theta_{-i}') - t_i(\theta_i', \theta_{-i}'') + t_i(\theta_i', \theta_{-i}')$.*

This binary relation orders transfers according to how increasing differences are. In differentiable environments, this definition is equivalent to saying that $\tilde{t} \succeq_{\text{ID}} t$, if and only if, for all $i \in N$ the cross partial derivatives of $\tilde{t}_i$ are larger than those of t_i, $\partial^2 \tilde{t}_i(\theta)/\partial\theta_i\partial\theta_j \geq \partial^2 t_i(\theta)/\partial\theta_i\partial\theta_j$, for all j and θ. This definition aims to capture the amount of complementarities contained in transfers and compares them accordingly. While relation $\succeq_{\text{ID}}$ is transitive and reflexive, it is not antisymmetric. Denote the set of $\succeq_{\text{ID}}$ equivalence classes of transfers by $\mathcal{T}$.[6]

In a supermodular game, the interval prediction is the interval between the largest and the smallest equilibrium. We compare supermodular mechanisms according to the size of the interval prediction of their induced game. The next proposition, taken from Mathevet (25), provides the tool to do so. If transfers t'' generate more complementarities than t', and if both induce truthtelling and supermodularity, then the equilibrium set induced by t'' includes that of t'.

For any $t \in \mathcal{T}$ such that $f = (x, t)$ is supermodular implementable, let $\overline{\theta}^t(\cdot)$ and $\underline{\theta}^t(\cdot)$ denote the largest and the smallest (Bayesian) equilibria of the induced game.

Proposition 1 *If (x, t'') and (x, t') are supermodular implementable social choice functions and if $t'' \succeq_{ID} t'$, then $[\underline{\theta}^{t'}(\cdot), \overline{\theta}^{t'}(\cdot)] \subset [\underline{\theta}^{t''}(\cdot), \overline{\theta}^{t''}(\cdot)]$.*

This proposition implies that the objective of minimizing the equilibrium set coincides with the objective of minimizing the complementarities. A social choice function $f = (x, t^*)$ will be minimally supermodular implementable if the transfers t^* elicit truthful revelation and induce a supermodular game with the weakest complementarities. This will give the tightest interval prediction around the truthful equilibrium.

4.2 Minimal Implementation under Total Orders

This section addresses minimal supermodular implementation within a class of transfers. We explicitly show how to convert any truthful mechanism into a supermodular mechanism while controlling for the intensity of the complementarities.

[6]Each equivalence class contains transfer functions t and $\tilde{t}$ such that $\tilde{t} \succeq_{\text{ID}} t$ and $t \succeq_{\text{ID}} \tilde{t}$ while $t \neq \tilde{t}$. Any quasi-order can be transformed into a partial order by using equivalence classes.

Our approach takes advantage of the totality of orders $\geq_i^1$. If the strategy sets are totally ordered, then the only requirement to check to satisfy Definition 1 is the increasing differences condition. Therefore, if the transfer functions ensure that (I) for each θ and i, $u_i^\Gamma(\hat{\theta}, \theta)$ has increasing differences in $(\hat{\theta}_i, \hat{\theta}_{-i})$ on $(\Theta_i, \geq_i^1) \times (\Theta_{-i}, \geq_{-i})$, then (II) the ex post game $\mathcal{G}(\theta)$ is supermodular for each θ, as desired. In this section, we restrict attention to the class of transfers that guarantee that (III) for each θ and i, $u_i^\Gamma(\hat{\theta}, \theta)$ has increasing differences in $(\hat{\theta}_i, \hat{\theta}_{-i})$ on $(\Theta_i, \geq_i^1) \times (\Theta_{-i}, \geq_i^2)$, where $\{(\geq_i^1, \geq_i^2)\}_i$ is a consistent profile of orders. Since $\geq_i^2$ is a linear extension of $\geq_{-i}$, (III) implies (I), hence (II) holds. Consider the following family of transfers:

Definition 4 *$\mathcal{F}(x, \{(\geq_i^1, \geq_i^2)\}_i)$ is the set of transfers $t \in \mathcal{T}$ such that (x, t) is truthfully implementable and $u_i^\Gamma(\hat{\theta}, \theta)$ has increasing differences on $(\Theta_i, \geq_i^1) \times (\Theta_{-i}, \geq_i^2)$ for each θ and i, where $\{(\geq_i^1, \geq_i^2)\}_i$ is consistent.*

We now define our concept of minimal supermodular implementation.

Definition 5 *A social choice function $f = (x, t^*)$ is minimally supermodular implementable over family $\mathcal{F}$ if it is supermodular implementable, $t^* \in \mathcal{F}$, and $t \succeq_{ID} t^*$ for all transfers $t \in \mathcal{F}$.*

Minimally supermodular transfers must elicit truthful revelation and produce the supermodular game with the weakest complementarities among $\mathcal{F}$. This gives the tightest equilibrium set within the class. Here is our first main result.

Theorem 1 *If $f = (x, t)$ is implementable, then for any consistent profile of orders $\{(\geq_i^1, \geq_i^2)\}_i$ there exist t^* such that (x, t^*) is minimally supermodular implementable over $\mathcal{F}(x, \{(\geq_i^1, \geq_i^2)\}_i)$.*

The theorem reaches a strong conclusion: for *any* implementable social choice function, its decision rule can be minimally supermodular implemented. There are no other restrictions on the decision rule or the valuation functions. Despite the finiteness of the type sets, there are infinitely many transfers that can supermodularly implement a decision rule for a given consistent profile of orders. Having a method for choosing the best among them is useful. To understand this, as well as our construction, start from any truth-revealing transfers $\{t_i\}$, and define

$$t_i^*(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]. \tag{4.1}$$

Transfers t_i^* satisfy $E_{\theta_{-i}}[t_i^*(\theta_i, \theta_{-i})] = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]$ for all θ_i, i.e. these two transfer functions have the same expected value when agents other than i report their type truthfully. Thus, if agent i finds it optimal to play truthfully under t_i (when others do so), then she also finds it optimal to do so under t_i^*. We conclude that for every collection of functions $\{\delta_i\}$, the transfers t^* also elicit truthful revelation. The problem becomes the choice of each δ_i, as there are infinitely many ways of inducing a supermodular game given a profile of orders. The proof provides an explicit formula for $\{\delta_i\}$ so that transfers t^* are the best within the family from the perspective of minimizing the interval prediction.

To sum up, our method suggests totally ordering type sets and then using our formula. Can this method be useful? In Section 2, it delivered a supermodular mechanism with a unique equilibrium, while ex post implementation was not an option. In the Appendix (see Section 7.1), it also delivers a supermodular mechanism with a unique equilibrium, while the original transfers produce multiple equilibria.

Given a choice of consistent orders, the theorem provides appropriate transfers. But there are many possible orders and the designer may want to discriminate among the many associated transfers. Suppose that the designer has a concept of distance, i.e. a metric d on Θ. Then Theorem 1 can be used to select the transfers that lead to the smallest equilibrium set across *all* the families. Let $\mathcal{F}^*(x)$ be the union of $\mathcal{F}(x, \{(\geq_i^1, \geq_i^2)\}_i)$ over all consistent orders $\{(\geq_i^1, \geq_i^2)\}_i$.

Corollary 1 *If $f = (x, t)$ is implementable, then there exist transfers t^{**} and consistent orders $\{(\geq_i^{*1}, \geq_i^{*2})\}_i$ such that (x, t^{**}) is minimally supermodular implementable over $\mathcal{F}(x, \{(\geq_i^{*1}, \geq_i^{*2})\}_i)$ and t^{**} give the smallest interval prediction in $\mathcal{F}^*(x)$ given d.*

Our corollary ultimately says that for every metric, there is a choice of total orders $(\geq_i^{*1}, \geq_i^{*2})$ for each i that is most adapted to d, since the equilibrium set resulting from the corresponding minimal transfers is minimized (under d) among all of $\mathcal{F}^*(x)$. The explanation is simple. For each profile of orders, the theorem provides the transfers that deliver the smallest interval prediction within the corresponding class. Since there are finitely many types, there are finitely many (consistent) profiles of orders. Therefore, there must be a profile of orders whose associated transfers deliver the smallest interval prediction under d among all of $\mathcal{F}^*(x)$.

4.3 Minimal Implementation with Order Reducibility

In this section, we study (unconditionally) minimal supermodular implementation by looking for the overall best transfers. In the previous section, the supermodular transfers were minimal within a class. We required that, for every agent i, increasing differences be satisfied on $(\Theta_i, \geq_i^1) \times (\Theta_{-i}, \geq_i^2)$. By doing so, we did not consider all the transfers that induce a supermodular game. In particular, some transfers may induce increasing differences on $(\Theta_i, \geq_i^1) \times (\Theta_{-i}, \geq_{-i})$ but not on the above product set, yet this is sufficient for our purpose. This is the case because $\geq_{-i}$ typically orders fewer elements than $\geq_i^2$, which affects the number of inequalities that have to hold to satisfy increasing differences. To summarize, our previous theorem was a conditional form of minimal supermodular implementation, while in this section, we aim for an unconditional form. In what follows, the order on Θ_{-i} is assumed to be the product order. For convenience, we write $V_i(x, \theta) = V_i(x_i, \theta)$ for all i to emphasize the dimension of the decision rule on which i's utility depends. This notation does not exclude allocative externalities, for an agent's own allocation x_i could be a function of another agent's allocation.

Definition 6 *A social choice function $f = (x, t^*)$ is minimally supermodular implementable if it is minimally supermodular implementable over family $\mathcal{T}$.*

We first show that the problem of finding minimally supermodular transfers is equivalent to solving a system of linear equations. This insight is highly useful, as it allows the application of standard methods and algorithms from numerical linear algebra (e.g. Paige and Saunders (29), Demmel (15)). In what follows, we refer to the *supermodularity* of a function $h_i : \Theta \to \mathbb{R}$ as the expression

$$h_i(\theta_i'', \theta_{-i}'') - h_i(\theta_i'', \theta_{-i}') - h_i(\theta_i', \theta_{-i}'') + h_i(\theta_i', \theta_{-i}')$$

where $\theta_i'' \geq_i^1 \theta_i'$ and $\theta_{-i}'' \geq_{-i} \theta_{-i}'$. Consider (4.1) and note that the supermodularity of t_i^* is equal to the supermodularity of δ_i. Therefore, our objective is to find a collection $\{\delta_i\}$ that induces increasing differences without introducing unnecessary complementarities. Before deriving the linear system, we define the concept of immediate successor/predecessor.

Definition 7 *For x' and x'' in a partially ordered set $(X, \geq_X)$, x'' is an immediate successor of x' (and x' is an immediate predecessor of x'') if (a) $x'' >_X x'$, and (b) the set $\{x \in X | x'' >_X x >_X x'\}$ is empty.*

A clear necessary condition for minimal supermodular implementation is that for all θ_i'', θ_i' where θ_i'' is an *immediate successor* of θ_i' in Θ_i, and for all θ_{-i}'', θ_{-i}', where θ_{-i}'' is an *immediate successor* of θ_{-i}' in Θ_{-i}, it holds that the supermodularity of δ_i (i.e. that of t_i^*) is equal to

$$-\min_{\theta \in \Theta}[V_i(x_i(\theta_i'', \theta_{-i}''), \theta) - V_i(x_i(\theta_i'', \theta_{-i}'), \theta) - V_i(x_i(\theta_i', \theta_{-i}''), \theta) + V_i(x_i(\theta_i', \theta_{-i}'), \theta)]. \quad (4.2)$$

If this equality were violated for some successive announcements, then we could build another transfer function that satisfies it for these announcements. Since the supermodularity of δ_i must always exceed (4.2) to induce a supermodular game, transfers

$\{t_i^*\}$ would not be the smallest under $\succeq_{\text{ID}}$. While necessity seems clear, it is not obvious that it suffices to search for $\{\delta_i\}$ whose supermodularity equals (4.2) for successive types only. Sufficiency comes from the proof of Theorem 1: the supermodularity of any function of two variables, when measured between non-successive elements, is equal to the sum of the supermodularities between all pairs of immediate successors in between. The intuition goes as follows. Take a function h with two variables, where each variable is in $\mathbb{N}$. Note that

$$h(2,3) - h(2,1) - (h(1,3) - h(1,1)) \tag{4.3}$$

is equal to

$$[h(2,3) - h(2,2) - (h(1,3) - h(1,2))] + [h(2,2) - h(2,1) - (h(1,2) - h(1,1))]. \tag{4.4}$$

The differences between non-successive types (1 and 3 are not immediate successors in (4.3)) are sums of differences between successive types, (4.4). Therefore, if the supermodularity of δ_i between successive types equals (4.2), which is the minimal requirement, then our previous observation implies that the supermodularity of δ_i between non-successive types must also be minimal. In conclusion, we just need to be concerned with supermodularity between successive types.

Given the above, we can view the problem of finding minimally supermodular transfers t_i^* as finding a vector δ_i that solves a system of linear equations $A \cdot \delta_i = b$. In this representation, δ_i is a column vector that contains the values of $\delta_i(\theta)$ for every $\theta \in \Theta$: $\delta_i = (\delta_i(\theta))$; A is a sparse matrix whose nonzero elements (four per row) are equal to -1 or 1, and positioned in a way that produces the supermodularity of δ_i (for successive types); b is a vector containing expressions (4.2), i.e. minima of valuation differences, between successive types. We give an example of this system in Section 7.2.

The next result provides necessary and sufficient conditions for the existence of minimally supermodular transfers. Such transfers exist if the condition described in following proposition holds for all player i.

Proposition 2 *(Farkas' lemma) The linear system $A \cdot \delta_i = b$ has a solution $\delta_i \geq 0$ if and only if, for every y, $A^T \cdot y \geq 0$ implies $b^T \cdot y \geq 0$.*

Assuming $\delta_i \geq 0$ is without loss of generality, because we can always add any positive constant c to any δ_i that solves the system and obtain another solution. The reason is that any constant gets canceled out when we form the supermodularity of a function. Theorem 2 has an intuitive interpretation, and it imposes a nice joint-condition on A and b. If we manipulate the supermodularities of the transfers by taking a linear combination y, and if this generates complementarities ($\delta_i^T A^T y \geq 0$), then it cannot be that the same linear combination of equations (4.2), i.e. the minimal valuations, generates strict complementarities ($b^T \cdot y < 0$). If this were the case, the transfers would add complementarities in places where there are already enough, hence they could not be minimally supermodular. The theorem is intuitive but the condition is impractical. Moreover, it establishes existence but does not provide explicit transfers. As already said, there exist useful techniques from numerical linear algebra to solve linear systems (e.g. Demmel (15) and Paige and Saunders (29)).

Nonetheless we provide a practical sufficient condition that ensures that minimal transfers exist and have a simple closed-form representation. To that end, we impose a richness condition on the decision rule.

Definition 8 *A decision rule $x(\theta)$ is order reducible if for each i, there are sets $\{G_p^i\}_{p=1}^P$ such that (a) $\Theta_{-i} = \cup_{p=1}^P G_p^i$, (b) for each θ_i, $x_i(\theta) = x_i(\theta_i, \theta'_{-i})$ for all $\theta_{-i}, \theta'_{-i} \in G_p^i$, and (c) if $\theta_{-i} \in G_p^i$, all immediate successors of θ_{-i} must be in $G_p^i \cup G_{p+1}^i$.*

Order reducibility ensures that, through the structure of the decision rule, opponents' type profiles can be put into groups to form a linear path between the images of x_i. This linear path preserves the product order on Θ_{-i} and does not impose any ordering of images between unordered types. To illustrate the definition, consider a setting with $n = 3$ agents and $\Theta_i = \{1, 2\}$ for all i. Assume types are ordered according to the usual order, i.e. $2 >_i^1 1$ for all i. Suppose the decision rule is $x_i(\theta) = x(\theta) = h(\sum \theta_i)$ where h is some strictly increasing real-valued function (Mathevet (25) presents several examples where the efficient decision rule takes this form). This decision rule is order reducible: for each agent i, it yields partition $G_1^i = \{(1, 1)\}$, $G_2^i = \{(1, 2), (2, 1)\}$ and $G_3^i = \{(2, 2)\}$. Note that for $n = 2$, order reducibility is trivially satisfied by all decision rules. Indeed, for each $j \neq i$, let each type in Θ_j form its own group with an index that corresponds to the position of the type under $>_j^1$. Below we present an example where order reducibility is violated.

Theorem 2 *Let $f = (x, t)$ be a social choice function such that x is order reducible. If f is implementable, then there exist t^o such that (x, t^o) is minimally supermodular implementable.*

This theorem establishes minimal supermodular implementability of a class of social choice functions. For any implementable social choice function, if the decision rule satisfies order reducibility, then there exist transfers t^o that guarantee truthful supermodular implementation as well as the smallest equilibrium set among all supermodular transfers. There are many ways in which a mechanism can be converted into a supermodular one. It is therefore useful to describe the best way to convert it (and when it exists) given the objective of minimized equilibrium set. In the proof of the theorem, we provide an explicit formula for transfers t^o.

Order reducibility may seem to be a restrictive condition. Unfortunately, relaxing it just a little in a simple setting already defies existence of minimal transfers, as the following example demonstrates. Consider a three-agent two-type example. Let $\Theta_i = \{1, 2\}$ and $2 >_i^1 1$ for all i. Choose a decision rule x such that for some i, the only possible grouping is $G_1^i = \{(1,1)\}$, $G_2^i = \{(1,2)\}$, $G_3^i = \{(2,1),(2,2)\}$ (actual group indexes do not matter). This decision rule is not order reducible since (2,1), despite being an immediate successor of (1,1), is in a group that does not immediately follow G_i^1. For most valuation functions, a solution to our system of linear equations does not exist in this case. Thus, our transfers t^o are not minimal but no other transfers are.

5 Uniqueness

In this section, we provide sufficient conditions for supermodular implementation in unique equilibrium. In light of our current results, a natural question to ask is: When does a minimal supermodular mechanism, i.e. one with the smallest equilibrium set, actually have a unique equilibrium? If a supermodular game has a unique equilibrium, then it is dominance-solvable, and many learning dynamics converge to the unique equilibrium (Milgrom and Roberts (27)). Supermodular implementation is, therefore, particularly appealing when truthtelling is the unique equilibrium. The study of unique supermodular implementation allows us to draw some conclusions regarding the type of environments — preferences and social choice functions — for which supermodular implementation may be most useful.

Recall that i's utility at type θ is denoted by $u_i^\Gamma(\hat\theta;\theta) = V_i(x(\hat\theta);\theta) + t_i(\hat\theta)$. At this point, the designer chooses a metric d_i to measure the distance between the elements of Θ_i for every i. For each i and θ, let $K_i(\theta)$ be a real number such that

$$\begin{aligned}(u_i^\Gamma(\theta_i'', \theta_{-i}''; \theta) - u_i^\Gamma(\theta_i', \theta_{-i}''; \theta)) - (u_i^\Gamma(\theta_i'', \theta_{-i}'; \theta) - u_i^\Gamma(\theta_i', \theta_{-i}'; \theta)) \\ \leq d_i(\theta_i'', \theta_i') K_i(\theta) \sum_{j \neq i} d_j(\theta_j'', \theta_j') \quad (5.1)\end{aligned}$$

for all $\theta_i'' \geq_i^1 \theta_i'$ and $\theta_{-i}'' \geq_{-i} \theta_{-i}'$. The rhs of (5.1) is an upper bound on the complementarities between i's own report and the other agents' reports given some mechanism. K_i exists because there are finitely many types. This number gauges the sensitivity of i's differential (or "marginal") utility to an increase in the other agents' reports and it is *endogenously* determined by the chosen transfers. Similarly, for each i and θ_{-i}, let

$\gamma_i(\theta_{-i})$ be a number such that

$$(V_i(x(\hat{\theta}_i'', \theta_{-i}); \theta_i'', \theta_{-i}) - V_i(x(\hat{\theta}_i', \theta_{-i}); \theta_i'', \theta_{-i})) - (V_i(x(\hat{\theta}_i'', \theta_{-i}); \theta_i', \theta_{-i}) - V_i(x(\hat{\theta}_i', \theta_{-i}); \theta_i', \theta_{-i})) \geq \gamma_i(\theta_{-i}) d_i(\hat{\theta}_i'', \hat{\theta}_i') d_i(\theta_i'', \theta_i') \quad (5.2)$$

for all $\hat{\theta}_i'' \geq_i^1 \hat{\theta}_i'$ and $\theta_i'' \geq_i^1 \theta_i'$. The rhs of equation (5.2) is a lower bound on the complementarities between i's own report and type when all other agents report truthfully. γ_i exists because there are finitely many types. This number represents a lower bound on the sensitivity of i's differential (or "marginal") valuation to an increase in i's own type and it is determined *exogenously* by the primitives of the model. In a differentiable environment, numbers K_i and γ_i would be bounds on cross-partial derivatives.

Intuitively, these numbers represent opposite forces regarding equilibrium multiplicity and uniqueness (Mathevet (26)). An agent who is extremely sensitive to her own type tends to make decisions independently of other agents. Large γ_i's, therefore, favor uniqueness. On the other hand, strong strategic complementarities connect agents together and create interdependence. Large K_i's, therefore, favor multiplicity.

In the next results, we formalize this trade-off into simple equations. Denote the truthful strategy by $\theta_i^T(\cdot)$. Use the standard notation for intervals, e.g. $(\theta_i, \theta_i^*) = \{\hat{\theta}_i : \theta_i <_i^1 \hat{\theta}_i <_i^1 \theta_i^*\}$. Strategy profiles are ordered by using the pointwise order, also denoted $\geq$ for simplicity: a strategy profile is larger than another if each player reports a larger type in the former than in the latter for every true type. Let $\bar{K}_i(\theta_i) = \max_{\theta_{-i}} K_i(\theta)$ for each i and θ_i.

Proposition 3 *Let f be a supermodular implementable social choice function. Choose any profile $\theta^*(\cdot) \geq \theta^T(\cdot)$. If there exist i, θ_i and $\hat{\theta}_i \in [\theta_i, \theta_i^*(\theta_i))$ such that*

$$\bar{K}_i(\theta_i) \sum_{j \neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j), \theta_j] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})] d_i(\hat{\theta}_i, \theta_i) < 0 \quad (5.3)$$

then $\theta^*(\cdot)$ is not a Bayesian equilibrium. The same conclusion applies to any profile $\theta^*(\cdot) \leq \theta^T(\cdot)$ if there exist i, θ_i and $\hat{\theta}_i \in (\theta_i^*(\theta_i), \theta_i]$ such that (5.3) holds.

The lhs of (5.3) summarizes the trade-off between opposite forces $\bar{K}_i$ and $E[\gamma_i(\cdot)]$. If a strategy profile $\theta^*(\cdot)$ is ordered w.r.t. truthtelling and if the uniqueness effect dominates, i.e. condition (5.3) holds, then this profile does not fall within the bounds of the interval prediction.[7]

[7]If the condition in the theorem holds, then it must hold when computed with the smallest possible $\bar{K}_i(\theta_i)$ and the largest possible $\gamma_i(\theta_{-i})$ (for each θ_{-i}). Therefore, using these "tightest" bounds for the computation would be a natural way for a designer to utilize this theorem.

This theorem cannot be useful for profiles such that $\theta_i^*(\theta_i)$ and θ_i are either equal or successive types for every i and θ_i. In that case, $d_i(\hat{\theta}_i, \theta_i)$ would be zero. Furthermore, although the theorem is useful to determine whether a given strategy profile is not an equilibrium, it does not allow a direct conclusion as to whether or not the mechanism has a unique equilibrium. The next proposition addresses this question.

Before proceeding, we define a measure of coarseness on agents' type spaces. For any type θ_i in Θ_i, letting θ_i' and θ_i'' be its immediate predecessor and immediate successor, we define

$$\varepsilon_i(\Theta_i) = \max_{\theta_i \in \Theta_i} \max\{d_i(\theta_i', \theta_i), d_i(\theta_i, \theta_i'')\}$$

to be a measure of the maximal distance between any type in Θ_i and its immediate successor or predecessor. As we get closer to the continuous case, $\varepsilon_i(\Theta_i) \to 0$. Define $\varepsilon(\Theta) = \max_i \varepsilon_i(\Theta_i)$ to be the overall measure of coarseness.

Our next result will be concerned with essential uniqueness. A profile $\theta^*(\cdot)$ is outside the neighborhood of truthtelling if $\theta^*(\cdot)$ and $\theta^T(\cdot)$ are strictly ordered (i.e. ordered and distinct), and if $E_{\theta_i}[d_i(\theta_i^*(\theta_i), \theta_i)] \geq E_{\theta_j}[d_j(\theta_j^*(\theta_j),\ \theta_j)]$ for all j implies that interval $(\theta_i, \theta_i^*(\theta_i))$ or $(\theta_i^*(\theta_i), \theta_i)$ — depending on which one is well defined — is nonempty for some θ_i. In words, a profile is outside the neighborhood of truthtelling if it is strictly larger or smaller than truthtelling, and if the agent that misreports the most on average has the option to report a non-truthful type in between truth θ_i and her actual report $\theta_i^*(\theta_i)$. In order for an agent to have this option, her original deception must be far enough from truthtelling, for otherwise the only possible deviation would be to report her type truthfully.

Definition 9 *The truthful equilibrium is essentially unique if any profile $\theta^*(\cdot)$ outside the neighborhood of truthtelling is not an equilibrium.*

According to essential uniqueness, only the profiles in the neighborhood of truth telling can be equilibria. As type sets become finer, and $\varepsilon(\Theta) \to 0$, the neighborhood of truthtelling collapses, and tends to be a single point.

Proposition 4 *Let f be a supermodular implementable social choice function (on Θ). If for every agent i*

$$(n-1)E_{\theta_i}[\bar{K}_i(\theta_i)] < E_{\theta_{-i}}[\gamma_i(\theta_{-i})],^{8} \tag{5.4}$$

then there is $\underline{\varepsilon}$ such that if $\varepsilon(\Theta) < \underline{\varepsilon}$, then the truthful equilibrium is essentially unique.

This proposition imposes two conditions, (5.4) and a richness requirement. Condition (5.4) says that the uniqueness effect dominates the multiplicity effect. While

[8] Bounds K_i and γ_i depend on the underlying type sets.

this captures the main driving force behind uniqueness, we must add the technical requirement that type sets be sufficiently fine. Otherwise, some untruthful profiles can become equilibria simply because some deviation is not available to an agent who would have otherwise chosen it. This proposition generalizes Mathevet (25)'s uniqueness result (Proposition 3, p.418) to our environments. In continuous type spaces, richness is obviously not an issue and only (5.4) matters.

For the sake of argument, assume that type sets are rich enough. At first the proposition seems to be mostly useful a posteriori. After the mechanism has been built, we can use it to check whether there is a unique equilibrium. However, it would be useful to know beforehand whether the design problem at hand is compatible with unique supermodular implementation given the primitives of the problem. Since our minimally supermodular transfers minimize the size of the equilibrium set, they are a natural choice for unique implementation. Moreover, they are entirely constructed from primitives of the model. Therefore, we can use them within Proposition 4 to obtain a condition that (i) can be checked before building the mechanism to (ii) determine whether unique supermodular implementation is attainable based on the primitives of the design problem. This allows us to draw conclusions about the type of environments for which supermodular implementation may be most useful.

For any implementable social choice function f, denote

$$K_i^*(\theta) = \max_{\{\theta_i'',\theta_i',\theta_{-i}'',\theta_{-i}'\}} \frac{V_i(\theta_i' \triangleright \theta_i'', \theta_{-i}''; \theta) - V_i(\theta_i' \triangleright \theta_i'', \theta_{-i}'; \theta) - H_i(\theta_i'', \theta_i', \theta_{-i}'', \theta_{-i}')}{d_i(\theta_i'', \theta_i') \sum_{j \neq i} d_j(\theta_j'', \theta_j')} \tag{5.5}$$

where $V_i(\theta_i' \triangleright \theta_i'', \cdot; \theta) = V_i(x_i(\theta_i'', \cdot); \theta) - V_i(x_i(\theta_i', \cdot); \theta)$ and H_i is the sum of elements

$$\min_{\theta \in \Theta} [V_i(\hat{\theta}_i' \triangleright \hat{\theta}_i'', \hat{\theta}_{-i}''; \theta) - V_i(\hat{\theta}_i' \triangleright \hat{\theta}_i'', \hat{\theta}_{-i}'; \theta)]$$

for all immediate successors $\hat{\theta}_i''$ and $\hat{\theta}_i'$, going from θ_i' to θ_i'', and all immediate successors $\hat{\theta}_{-i}''$ and $\hat{\theta}_{-i}'$, going from θ_{-i}' to θ_{-i}''. Define $\bar{K}_i^*(\theta_i) = \max_{\theta_{-i}} K_i^*(\theta)$. Given (5.5), note that the value of $\bar{K}_i^*$, which bounds the degree of complementarities under the minimally supermodular transfers, depends only on the primitives of the model. When the designer uses the minimally supermodular transfers, $\bar{K}_i^*$ is the value that appears in condition (5.4). Hence, when computed with $\bar{K}_i^*$ as given by (5.5), inequality (5.4) becomes a condition involving only the primitives of the model. If this inequality holds, then supermodular implementation is particularly well-suited for the design problem at hand, because the minimal transfers supermodularly implement the social choice function and truthtelling is essentially unique.

Condition (5.5) has a nice interpretation. It measures how much the supermodularity of the valuations vary across true types. We know that the designer must induce a supermodular game for any realization of types.[9] A large $\bar{K}_i^*$ is caused by valuation functions that produce large substitute effects at some types (say θ) and large complementarities at others (say θ'). Since the designer does not know the realization of the true type, she will need to add a lot of complementarities through the transfers to ensure that the game is supermodular at θ. However, this may induce a game that is "too supermodular" at θ' as there are already enough complementarities at that type, which may cause multiplicity.

6 Conclusion

This paper extends supermodular mechanism design to environments with interdependent valuations, informational and allocative externalities, and arbitrary finite type spaces. While realistic, these environments present a serious challenge to mechanism designers. It is typically impossible to employ dominant strategy and ex post equilibrium. This makes Bayesian implementation particularly relevant. In this context, supermodular Bayesian mechanisms are attractive.

The main motivation behind our mechanism design approach is to facilitate convergence to a desired equilibrium. This includes two problems: the robustness to bounded rationality (i.e. learning) and the multiple equilibrium problem. Supermodular mechanisms have nice learning properties, and the interval between their extremal equilibria contains all the limit points of learning dynamics. To some extent, this interval "measures" the multiple equilibrium problem. Our methodology is to impose orders on type sets, and given these orders, to induce a supermodular mechanism and to minimize its interval prediction by weakening the complementarities. It is worth mentioning that agents need not be aware of these orders. While the analyst can exploit the monotonicity of agents' best responses to derive convergence properties, agents need not know, or be informed, that their best responses are monotonic. These orders are just a tool for the designer. As a whole, our concerns have focused on behavioral robustness and left other issues unanswered.

First, our mechanisms are parametric. The designer needs to know the common prior beliefs to construct the mechanisms, which is demanding (Ledyard (23)). Moreover, small mistakes with respect to prior beliefs might lead to shifts in equilibrium behavior

[9] It is sufficient but not necessary that the ex post game be supermodular for each realization in order for the ex ante Bayesian game to be supermodular. For example, if the prior is mostly concentrated on some subset Θ' of Θ, it may not be necessary to make the ex post payoffs supermodular for types in $\Theta \backslash \Theta'$. Of course, the possibility of neglecting $\Theta \backslash \Theta'$ depends on how unlikely that set is compared to how submodular the utility function may be for types in that set.

and deviations from efficiency. Along this line, the literature on robust mechanism design (e.g. Bergemann and Morris (3)) advocates the use of ex post equilibrium. However, as we have already said, this is not always possible in our environments.

Second, we have omitted the issue of budget balancing. Robustness to bounded rationality may well come at the price of a balanced budget, i.e. full efficiency. In Section 7.1, we presented an example where starting from balanced transfers and multiple equilibria, the designer could achieve dominance-solvability, hence uniqueness, and allocation efficiency but our transfers were not balanced. Reconciling budget balancing and minimal supermodularity (or, in general, dominance solvability) would be optimal but this is an open question. If both properties were exclusive in general, the designer would be faced with a difficult choice: balancing budget at the price of the implementation target (in case players do not learn to play the desired equilibrium), or guaranteeing the implementation target is reached at the price of a balanced budget. Further research is needed to shed light on the eventual tension between robustness and full efficiency.

7 Appendix

7.1 Another Motivating Example

Consider the motivating example of Section 2. The designer may choose the following balanced transfers to implement the efficient decision rule:

$t_1(\hat{\theta})$	(2, 1)	(1, 2)
$(2,1)$	.12	.10
$(1,2)$	.01	.35

$t_2(\hat{\theta})$	(2, 1)	(1, 2)
$(2,1)$	-.12	-.10
$(1,2)$	-.01	-.35

Given these transfers, the resulting payoff matrix for the ex ante Bayesian game is

EU	truthtelling	constant (2,1)	constant (1,2)	always lie
truthtelling	.14*; 0*	.1; -.04	.2; -.1	.2; -.2
constant (2,1)	.1; -.1	.1*;0*	.1; -.1	.1; -.04
constant (1,2)	0;.02*	0; -.01	.4*; -.2	.3*; -.3
always lie	0; 0	0; .04*	.3; -.2	.2;-.1

Both truthtelling and a constant announcement of (2,1) by both players are Bayesian equilibria. If we instead use the supermodular transfers that add minimal complementarities

$t_1^{SM}(\hat{\theta})$	(2, 1)	(1, 2)
$(2,1)$	.1	.1
$(1,2)$	.02	.2

$t_2^{SM}(\hat{\theta})$	(2, 1)	(1, 2)
$(2,1)$	-.05	-.4
$(1,2)$	-.05	-.2

we obtain the ex ante payoff matrix:

EU	truthtelling	constant (2,1)	constant (1,2)	always lie
truthtelling	.14*; 0*	.1;* -.05	.1; -.1	.1; -.2
constant (2,1)	.13; 0	.1; .1*	.1; -.4	.1; -.3
constant (1,2)	.04; 0*	0; -.05	.2*; -.1	.2*;-.1
always lie	0; 0	.02 .1*	.2; -.3	.2; -.2

Converting the original mechanism into a minimally supermodular mechanism has solved the multiple equilibrium problem. Truthtelling is the unique Bayesian equilibrium.

7.2 An Example of Linear System for Minimal Supermodular Implementation

Consider a setting with $n = 3$ agents, and types in $\Theta_i = \{1, 2\}$ for all i. Assume the conventional order $2 >_i^1 1$ for all i. For each player i, in order to minimally supermodular implement the decision rule x, we are interested in finding a solution to the following system of linear equations:

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \delta_i(1,(1,1)) \\ \delta_i(2,(1,1)) \\ \delta_i(1,(1,2)) \\ \delta_i(2,(1,2)) \\ \delta_i(1,(2,1)) \\ \delta_i(2,(2,1)) \\ \delta_i(1,(2,2)) \\ \delta_i(2,(2,2)) \end{pmatrix} = \begin{pmatrix} -\min_\theta X(\theta) \\ -\min_\theta Y(\theta) \\ -\min_\theta Z(\theta) \\ -\min_\theta W(\theta) \end{pmatrix}.$$

where

$$\begin{aligned} X(\theta) &= V_i(x_i(2,(1,2));\theta) - V_i(x_i(1,(1,2));\theta) - V_i(x_i(2,(1,1));\theta) + V_i(x_i(1,(1,1));\theta) \\ Y(\theta) &= V_i(x_i(2,(2,1));\theta) - V_i(x_i(1,(2,1));\theta) - V_i(x_i(2,(1,1));\theta) + V_i(x_i(1,(1,1));\theta) \\ Z(\theta) &= V_i(x_i(2,(2,2));\theta) - V_i(x_i(1,(2,2));\theta) - V_i(x_i(2,(1,2));\theta) + V_i(x_i(1,(1,2));\theta) \\ W(\theta) &= V_i(x_i(2,(2,2));\theta) - V_i(x_i(1,(2,2));\theta) - V_i(x_i(2,(2,1));\theta) + V_i(x_i(1,(2,1));\theta). \end{aligned}$$

Consider agent $i \in N$, whose valuations are given by:

$V_i(\cdot;\theta)$	$(1,1,1)$	(1,1,2)	(1,2,1)	(2,1,1)	(1,2,2)	(2,1,2)	(2,2,1)	(2,2,2)
A	0	0	0	0	0	0	0	0
B	3	1	2	1	2	2	2	2

Let us assume the decision rule to be implemented is:

	$(1,1,1)$	(1,1,2)	(1,2,1)	(2,1,1)	(1,2,2)	(2,1,2)	(2,2,1)	(2,2,2)
$\tilde{x}_i(\theta)$	B	A	A	B	B	B	B	B

Then the rhs of the system becomes:

$$\begin{pmatrix} -\min_\theta X \\ -\min_\theta Y \\ -\min_\theta Z \\ -\min_\theta W \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 3 \\ 3 \end{pmatrix}$$

One possible solution for the system is $\delta_i = (0,1,0,0,1,1,0,3)^T$.

7.3 Proofs

Proof of Theorem 1 Take a consistent profile of orders $\{(\geq_i^1, \geq_i^2)\}_i$. For every $i \in N$, each element $\theta_i \in \Theta_i$ is assigned an index k that corresponds to its position in the set Θ_i under the total order $\geq_i^1$. Similarly, each element $\theta_{-i} \in \Theta_{-i}$ is assigned an index q according to the total order order $\geq_i^2$ on Θ_{-i}. Suppose that $f = (x, t)$ is implementable. Letting

$$\delta_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q) \equiv -\sum_{l=1}^{k-1}\sum_{z=1}^{q-1} \min_{\theta\in\Theta}[V_i(x(\hat{\theta}_i^{l+1}, \hat{\theta}_{-i}^{z+1}); \theta) - V_i(x(\hat{\theta}_i^{l}, \hat{\theta}_{-i}^{z+1}); \theta) - V_i(x(\hat{\theta}_i^{l+1}, \hat{\theta}_{-i}^{z}); \theta) + V_i(x(\hat{\theta}_i^{l}, \hat{\theta}_{-i}^{z}); \theta)]. \quad (7.1)$$

for all $\hat{\theta}_i^k \in \Theta_i$ and $\hat{\theta}_{-i}^q \in \Theta_{-i}$, we define

$$t_i^*(\hat{\theta}_i^k, \hat{\theta}_{-i}^q) \equiv \delta_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i^k, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_i^k, \theta_{-i})] \quad (7.2)$$

and show that (x, t^*) is minimally supermodular implementable.

Step 1. We show that t_i^* has smaller one-step supermodularity than any t_i such that (x, t) is supermodular implementable.

Let us define the one-step supermodularity of $V_i(x(\cdot); \theta)$ at any given announcement $(\hat{\theta}_i^k, \hat{\theta}_{-i}^q)$ as

$$g_i(k, q; \theta) \equiv V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q+1}); \theta) - V_i(x(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+1}); \theta) - V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q}); \theta) + V_i(x(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}); \theta). \quad (7.3)$$

For notational simplicity, we define

$$\begin{aligned} d_i(k, q) &\equiv \min_{\theta\in\Theta}[V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q+1}); \theta) - V_i(x(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+1}); \theta) \\ &\quad -V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q}); \theta) + V_i(x(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}); \theta)] \\ &= \min_{\theta\in\Theta} g_i(k, q; \theta). \end{aligned} \quad (7.4)$$

Since the one-step supermodularity of t_i^* is equivalent to the one-step supermodularity of δ_i we have

$$\begin{aligned} s_i(k, q) &\equiv \delta_i(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q+1}) - \delta_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+1}) - \delta_i(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q}) + \delta_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}) \\ &= -\sum_{l=1}^{k}\sum_{z=1}^{q} d_i(l, z) + \sum_{l=1}^{k-1}\sum_{z=1}^{q} d_i(l, z) + \sum_{l=1}^{k}\sum_{z=1}^{q-1} d_i(l, z) - \sum_{l=1}^{k-1}\sum_{z=1}^{q-1} d_i(l, z) \\ &= -d_i(k, q) \end{aligned} \quad (7.5)$$

as the one-step supermodularity of t_i^* (and δ_i).

Therefore, the one-step supermodularity of $(V_i + t_i^*)$ is given by

$$g_i(k, q; \theta) + s_i(k, q) \geq 0 \tag{7.6}$$

for all $\hat{\theta}_i^k, \hat{\theta}_{-i}^q$, θ, k, q, and i.

Denote the one-step supermodularity of transfer t_i as $sm_1(t_i; k, q)$, that is:

$$sm_1(t_i; k, q) = t_i(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q+1}) - t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^{q+1}) - t_i(\hat{\theta}_i^{k+1}\hat{\theta}_{-i}^q) + t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q).$$

For all transfers t such that (x, t) is supermodular implementable, it must hold that $g_i(k, q; \theta) + sm_1(t_i; k, q) \geq 0$ for all $\theta \in \Theta$, which is equivalent to:

$$\begin{aligned} sm_1(t_i; k, q) \geq -\min_{\theta \in \Theta} [&V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q+1}); \theta) - V_i(x(\hat{\theta}_i^k, \hat{\theta}_{-i}^{q+1}); \theta) \\ &- V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^q); \theta) + V_i(x(\hat{\theta}_i^k, \hat{\theta}_{-i}^q); \theta)] = s_i(k, q). \end{aligned} \tag{7.7}$$

The above shows that if (x, t) is supermodular implementable then the one-step supermodularity of transfers t is necessarily (weakly) greater than the one-step supermodularity of transfers t^*, which establishes Step 1.

Step 2. We show that the (multiple-step) supermodularity of any function of two variables is a sum of one-step supermodularities. Let us define the "(η, γ)-step supermodularity" of any function $t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q)$ as

$$SM_{(\eta,\gamma)}(t_i; k, q) = t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^q) + t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q). \tag{7.8}$$

Note that

$$\begin{aligned} t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma}) = sm_1(t_i; k+\eta-1, q+\gamma-1) &+ t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma}) \\ &+ t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma-1}), \end{aligned} \tag{7.9}$$

and so it follows from (7.8) that

$$\begin{aligned} SM_{(\eta,\gamma)}(t_i; k, q) = \big[&sm_1(t_i; k+\eta-1, q+\gamma-1) + t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma}) + t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) \\ &- t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma-1})\big] - t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^q) + t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q)). \end{aligned} \tag{7.10}$$

Note that

$$t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma}) = sm_1(t_i; k+\eta-2, q+\gamma-1) + t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma})$$
$$+ t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma-1}), \quad (7.11)$$

and therefore it follows from (7.10) that

$$SM_{(\eta,\gamma)}(t_i; k, q) = sm_1(t_i; k+\eta-1, q+\gamma-1) + \Big[sm_1(t_i; k+\eta-2, g+\gamma-1)$$
$$+ t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma}) + t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma-1})\Big]$$
$$+ t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q}) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}) \quad (7.12)$$

which is equal to

$$\sum_{n=1}^{2} sm_1(t_i; k+\eta-n, q+\gamma-1) + t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma-1})$$
$$+ t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q}) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}). \quad (7.13)$$

Proceeding iteratively with this process of substitution and regrouping of terms for $n = (1, \ldots, \eta)$ we obtain

$$\begin{aligned}
SM_{(\eta,\gamma)}(t_i; k, q) &= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-1) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma-1}) \\
&\quad + t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q}) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}) \\
&= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-1) \\
&\quad + t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q}) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}) \\
&= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-1) + SM_{(\eta,\gamma-1)}(t_i; k, q). \qquad (7.14)
\end{aligned}$$

Iterrating on Equation (7.14) for $m = 1, \dots, \gamma - 1$ we obtain:

$$
\begin{aligned}
SM_{(\eta,\gamma)}(t_i; k, q) &= \sum_{n=1}^{\eta} sm_1(t_i; k + \eta - n, q + \gamma - 1) + SM_{(\eta,\gamma-1)}(t_i; k, q) \\
&= \sum_{n=1}^{\eta} sm_1(t_i; k + \eta - n, q + \gamma - 1) + \sum_{n=1}^{\eta} sm_1(t_i; k + \eta - n, q + \gamma - 2) + SM_{(\eta,\gamma-2)}(t_i; k, q) \\
&= \sum_{n=1}^{\eta} \sum_{m=1}^{\gamma-1} sm_1(t_i; k + \eta - n, q + \gamma - m) + SM_{(\eta,1)}(t_i; k, q). \qquad (7.15)
\end{aligned}
$$

Now, using the fact that

$$
\begin{aligned}
SM_{(\eta,1)}(k, q) &= t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+1}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+1}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q}) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}) \\
&= sm_1(t_i; k + \eta - 1, q) + SM_{(\eta-1,1)}(t_i; k, q) \\
&= \textstyle\sum_{n=1}^{\eta} sm_1(t_i; k + \eta - n, q)
\end{aligned}
$$

and plugging this into Equation (7.15) we obtain

$$
\begin{aligned}
SM_{(\eta,\gamma)}(t_i; k, q) &= \sum_{n=1}^{\eta} \sum_{m=1}^{\gamma-1} sm_1(t_i; k + \eta - n, q + \gamma - m) + \sum_{n=1}^{\eta} sm_1(t_i; k + \eta - n, q) \\
&= \sum_{n=1}^{\eta} \sum_{m=1}^{\gamma} sm_1(t_i; k + \eta - n, q + \gamma - m) \\
&= \sum_{l=k}^{k+\eta-1} \sum_{z=q}^{q+\gamma-1} sm_1(t_i; l, z). \qquad (7.16)
\end{aligned}
$$

Thus, the multiple-step supermodularity of any function of two ordered variables is equal to the sum of one-step supermodularities, which establishes Step 2.

Step 3. Conclusion. Note that

$$
E_{\theta_{-i}}[t_i^*(\hat{\theta}_i^k, \theta_{-i})] = E_{\theta_{-i}}[\delta_i(\hat{\theta}_i^k, \theta_{-i})] - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i^k, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_i^k, \theta_{-i})] = E_{\theta_{-i}}[t_i(\hat{\theta}_i^k, \theta_{-i})] \qquad (7.17)
$$

and therefore transfers t_i and t_i^* have the same expected value given that all other agents report their types truthfully. That is, assuming truthful reporting, the expected utility of an agent is the same under t_i and t_i^*. Since (x, t) is truthfully implementable, the above implies that (x, t^*) is also truthfully implementable.

Using the result eshablished in Step 2, the (η, γ)-step supermodularity of $V_i(x(\cdot); \theta)$

at any given announcement $(\hat{\theta}_i^k, \hat{\theta}_{-i}^q)$ can now be written as:

$$\begin{aligned} G_i^{(\eta,\gamma)}(k,q;\theta) &= V_i(x(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma});\theta) - V_i(x(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma});\theta) \\ &\quad - V_i(x(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q});\theta) + V_i(x(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q});\theta) \\ &= \sum_{l=k}^{k+\eta-1} \sum_{z=q}^{q+\gamma-1} g_i(l,z;\theta). \end{aligned} \tag{7.18}$$

and the (η,γ)-step supermodularity of t_i^* is analogously given by

$$\begin{aligned} S_i^{(\eta,\gamma)}(k,q) &= \delta_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma}) - \delta_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma}) - \delta_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q}) + \delta_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}) \\ &= -\sum_{l=1}^{k+\eta-1} \sum_{z=1}^{q+\gamma-1} d_i(l,z) + \sum_{l=1}^{k-1} \sum_{z=1}^{q+\gamma-1} d_i(l,z) + \sum_{l=1}^{k+\eta-1} \sum_{z=1}^{q-1} d_i(l,z) - \sum_{l=1}^{k-1} \sum_{z=1}^{q-1} d_i(l,z) \\ &= -\sum_{l=k}^{k+\eta-1} \sum_{z=q}^{q+\gamma-1} d_i(l,z). \end{aligned} \tag{7.19}$$

It is straightforward to check that $G_i^{(\eta,\gamma)}(k,q;\theta)+S_i^{(\eta,\gamma)}(k,q) \geq 0$ for all $\hat{\theta}_i^k, \hat{\theta}_{-i}^q, \theta, k, q, \eta, \gamma$ and i and, therefore, t^* is supermodular implementable.

Moreover, Step 1 says that t^* has the smallest one-step supermodularity among all supermodular transfers t. Combined with Step 2, this establishes that t^* has the smallest (η,γ)-step supermodularity for any (η,γ) among all supermodular transfers t. Thus we conclude that (x,t^*) is minimally supermodular implementable under the chosen order profile $\{(\geq_i^1, \geq_i^2)\}_i$. Q.E.D

Proof of Corollary 1 In the proof of Theorem 1, we constructed transfers that minimally supermodular implemented the decision rule x under some chosen consistent profile of orders $\{(\geq_i^1, \geq_i^2)\}_i$. Each $(\geq_i^1, \geq_i^2)$ is a pair of complete orders on finite sets. Since there are finitely many agents, for each i there are finitely many complete orders, and consequently, finitely many consistent profiles. For each such profile, we can compute the distance between the largest and the smallest equilibrium in the ex ante induced game under the minimal transfers, using a metric d. Among all consistent profiles of orders we can thus choose the one associated with the smallest interval prediction as measured by d: denote this profile of orders by $\{(\geq_i^{*1}, \geq_i^{*2})\}_i$ and the corresponding minimal transfers by t^{**}. Therefore, t^{**} give the smallest interval prediction under d among all minimally supermodular transfers on consistent profiles of orders. Q.E.D

Proof of Theorem 2 Suppose $f = (x,t)$ is implementable and x is order reducible. For every $i \in N$, assign each element $\theta_i \in \Theta_i$ an index k that corresponds to its position in the set Θ_i under the total order $\geq_i^1$. Since x is order reducible, each element $\theta_{-i} \in \Theta_{-i}$ can be assigned an index p according to the group G_p^i to which it belongs. That is, more than one element θ_{-i} can be assigned the same index p, as all the elements in group G_p share the same index p. Letting

$$\delta_i(\theta_i^k, \theta_{-i}^p) = -\sum_{l=1}^{k-1}\sum_{z=1}^{p-1} \min_{\theta\in\Theta}[V_i(x(\theta_i^{l+1}, \theta_{-i}^{z+1}), \theta) - V_i(x(\theta_i^{l}, \theta_{-i}^{z+1}), \theta) - V_i(x(\theta_i^{l+1}, \theta_{-i}^{z}), \theta) + V_i(x(\theta_i^{l}, \theta_{-i}^{z}), \theta)] \tag{7.20}$$

for all $\theta_i^k \in \Theta_i$ and $\theta_{-i}^p \in \Theta_{-i}$, we define

$$t_i^o(\theta_i^k, \theta_{-i}^p) = \delta_i(\theta_i^k, \theta_{-i}^p) - E_{\theta_{-i}}[\delta_i(\theta_i^k, \theta_{-i})] + E_{\theta_{-i}}[t_i(\theta_i^k, \theta_{-i})] \tag{7.21}$$

and show that (x, t^o) is minimally supermodular implementable.

Note that $E_{\theta_{-i}}[t_i^o(\theta_i^k, \theta_{-i})] = E_{\theta_{-i}}[t_i(\theta_i^k, \theta_{-i})]$ and thus (x, t^o) is truthfully implementable. Moreover, the supermodularity of $t_i^o(\theta_i^k, \theta_{-i}^p)$ is equal to the supermodularity of $\delta_i(\theta_i^k, \theta_{-i}^p)$. We proceed to show in separate steps of the proof that transfers t^o achieve minimal supermodularities across immediate successors on $(\Theta_i, \geq_i^1)$ and $(\Theta_{-i}, \geq_{-i})$ (Step 1) and that the supermodularities of t_i^o across (multiple-step) successive types are sums of supermodularities between immediate (one-step) successors (Step 2).

Step 1. Consider any two pairs of immediate successors $\theta_i'' \geq_i^1 \theta_i'$ and $\theta_{-i}'' \geq_{-i} \theta_{-i}'$. As they are immediate successors, we can instead write $\theta_i^{k+1} \geq_i^1 \theta_i^k$. The (one-step) supermodularity of t_i^o is

$$t_i^o(\theta_i^{k+1}, \theta_{-i}'') - t_i^o(\theta_i^{k}, \theta_{-i}'') - t_i^o(\theta_i^{k+1}, \theta_{-i}') + t_i^o(\theta_i^{k}, \theta_{-i}') = \delta_i(\theta_i^{k+1}, \theta_{-i}'') - \delta_i(\theta_i^{k}, \theta_{-i}'') - \delta_i(\theta_i^{k+1}, \theta_{-i}') + \delta_i(\theta_i^{k}, \theta_{-i}'). \tag{7.22}$$

Since x is order reducible and $\theta_{-i}'' \geq_{-i} \theta_{-i}'$ are immediate successors, it must be that either $\theta_{-i}', \theta_{-i}'' \in G_p^i$ or $\theta_{-i}' \in G_p^i$ and $\theta_{-i}'' \in G_{p+1}^i$.

Case 1. If $\theta_{-i}', \theta_{-i}'' \in G_p^i$, then by order reducibility, $x(\theta_i, \theta_{-i}') = x(\theta_i, \theta_{-i}'')$ for all θ_i and we obtain

$$V_i(x(\theta_i^{k+1}, \theta_{-i}''); \theta) - V_i(x(\theta_i^{k}, \theta_{-i}''); \theta) - V_i(x(\theta_i^{k+1}, \theta_{-i}'); \theta) + V_i(x(\theta_i^{k}, \theta_{-i}'); \theta) = 0. \tag{7.23}$$

Using equation (7.20) for δ_i we have that $\delta_i(\theta_i, \theta_{-i}') = \delta_i(\theta_i, \theta_{-i}'') = \delta_i(\theta_i, \theta_{-i}^p)$ for all θ_i.

The supermodularity of t_i^o hence becomes:

$$t_i^o(\theta_i^{k+1}, \theta''_{-i}) - t_i^o(\theta_i^k, \theta''_{-i}) - t_i^o(\theta_i^{k+1}, \theta'_{-i}) + t_i^o(\theta_i^k, \theta'_{-i}) = \\ \delta_i(\theta_i^{k+1}, \theta_{-i}^p) - \delta_i(\theta_i^k, \theta_{-i}^p) - \delta_i(\theta_i^{k+1}, \theta_{-i}^p) + \delta_i(\theta_i^k, \theta_{-i}^p) = 0. \quad (7.24)$$

Hence, for all t_i such that (x, t) is supermodular implementable it must hold that:

$$t_i(\theta_i^{k+1}, \theta''_{-i}) - t_i(\theta_i^k, \theta''_{-i}) - t_i(\theta_i^{k+1}, \theta'_{-i}) + t_i(\theta_i^k, \theta'_{-i}) \geq \\ - \min_\theta [V_i(x(\theta_i^{k+1}, \theta''_{-i}); \theta) - V_i(x(\theta_i^k, \theta''_{-i}); \theta) - V_i(x(\theta_i^{k+1}, \theta'_{-i}); \theta) + V_i(x(\theta_i^k, \theta'_{-i}); \theta)] \\ = 0 = t_i^o(\theta_i^{k+1}, \theta''_{-i}) - t_i^o(\theta_i^k, \theta''_{-i}) - t_i^o(\theta_i^{k+1}, \theta'_{-i}) + t_i^o(\theta_i^k, \theta'_{-i}). \quad (7.25)$$

Therefore, for all i and immediate successors $\theta'_{-i}, \theta''_{-i} \in G_p^i$, transfers t_i^o have the smallest one-step supermodularity.

<u>Case 2.</u> If $\theta'_{-i} \in G_p^i$ and $\theta''_{-i} \in G_{p+1}^i$, using equation (7.20) to obtain the supermodularity of t_i^o we get

$$t_i^o(\theta_i^{k+1}, \theta''_{-i}) - t_i^o(\theta_i^k, \theta''_{-i}) - t_i^o(\theta_i^{k+1}, \theta'_{-i}) + t_i^o(\theta_i^k, \theta'_{-i}) = \\ \delta_i(\theta_i^{k+1}, \theta_{-i}^{p+1}) - \delta_i(\theta_i^k, \theta_{-i}^{p+1}) - \delta_i(\theta_i^{k+1}, \theta_{-i}^p) + \delta_i(\theta_i^k, \theta_{-i}^p) = \\ - \min_\theta [V_i(x(\theta_i^{k+1}, \theta_{-i}^{p+1}); \theta) - V_i(x(\theta_i^k, \theta_{-i}^{p+1}); \theta) - V_i(x(\theta_i^{k+1}, \theta_{-i}^p); \theta) + V_i(x(\theta_i^k, \theta_{-i}^p); \theta)] = \\ - \min_\theta [V_i(x(\theta_i^{k+1}, \theta''_{-i}); \theta) - V_i(x(\theta_i^k, \theta''_{-i}); \theta) - V_i(x(\theta_i^{k+1}, \theta'_{-i}); \theta) + V_i(x(\theta_i^k, \theta'_{-i}); \theta)]. \\ (7.26)$$

Hence, for all t_i such that (x, t) is supermodular implementable it must hold that:

$$t_i(\theta_i^{k+1}, \theta''_{-i}) - t_i(\theta_i^k, \theta''_{-i}) - t_i(\theta_i^{k+1}, \theta'_{-i}) + t_i(\theta_i^k, \theta'_{-i}) \geq \\ - \min_\theta [V_i(x(\theta_i^{k+1}, \theta''_{-i}); \theta) - V_i(x(\theta_i^k, \theta''_{-i}); \theta) - V_i(x(\theta_i^{k+1}, \theta'_{-i}); \theta) + V_i(x(\theta_i^k, \theta'_{-i}); \theta)] \\ = t_i^o(\theta_i^{k+1}, \theta''_{-i}) - t_i^o(\theta_i^k, \theta''_{-i}) - t_i^o(\theta_i^{k+1}, \theta'_{-i}) + t_i^o(\theta_i^k, \theta'_{-i}). \quad (7.27)$$

Therefore, for all i and immediate successors $\theta'_{-i} \in G_p^i$ and $\theta''_{-i} \in G_{p+1}^i$, transfers t_i^o have the smallest one-step supermodularity.

Cases 1 and 2 allow us to conclude that transfers t^o achieve minimal supermodularities across any pair of immediate successors on $(\Theta_i, \geq_i^1)$ and $(\Theta_{-i}, \geq_{-i})$, as long as x is order reducible.

Step 2. Consider the supermodularity between successive types θ_i^k, θ_i^{k+q} and $\theta'_{-i} \in G_p^i$, $\theta''_{-i} \in G_{p+m}^i$. For $q = 1$ and $m = 1$ (or $m = 0$) this would reduce to the case of supermodularities between immediate successors considered in Step 1. Using equation

(7.20), we obtain

$$
\begin{aligned}
t_i^o(\theta_i^{k+q},\theta''_{-i}) - t_i^o(\theta_i^k,\theta''_{-i}) - t_i^o(\theta_i^{k+q},\theta'_{-i}) + t_i^o(\theta_i^k,\theta'_{-i}) = \\
\delta_i(\theta_i^{k+q},\theta_{-i}^{p+m}) - \delta_i(\theta_i^k,\theta_{-i}^{p+m}) - \delta_i(\theta_i^{k+q},\theta_{-i}^p) + \delta_i(\theta_i^k,\theta_{-i}^p) = \\
-\sum_{l=k}^{k+q-1}\sum_{z=p}^{p+m-1}\min_{\theta\in\Theta}[V_i(x(\theta_i^{l+1},\theta_{-i}^{z+1}),\theta) - V_i(x(\theta_i^l,\theta_{-i}^{z+1}),\theta) \\
- V_i(x(\theta_i^{l+1},\theta_{-i}^z),\theta) + V_i(x(\theta_i^l,\theta_{-i}^z),\theta)]. \quad (7.28)
\end{aligned}
$$

Hence, the q,m-step supermodularity of transfers t_i^o is a sum of all the one-step supermodularities between the groups G_p and G_{p+m}. We next show that this sum between the groups is equivalent to a sum of minimal one-step supermodularities on $\Theta_i \times \Theta_{-i}$, all of which need to be minimized for minimal supermodular implementation to hold.

Take a sequence $\theta_i^k,\ldots,\theta_i^{k+q}$ of immediate successors under $\geq_i^1$, and a sequence $\theta_{-i}^1,\ldots,\theta_{-i}^{1+s}$ of immediate successors under $\geq_{-i}$ such that $\theta_{-i}^1 = \theta'_{-i}$ and $\theta_{-i}^{1+s} = \theta''_{-i}$. Since $\theta'_{-i} \in G_p^i$, $\theta''_{-i} \in G_{p+m}^i$, and x is order reducible, it cannot be that θ''_{-i} is more that s groups away from θ'_{-i}, i.e. it must be that $s \geq m$.

<u>Case 1.</u> If $m = s$, then

$$
\begin{aligned}
t_i^o(\theta_i^{k+q},\theta''_{-i}) - t_i^o(\theta_i^k,\theta''_{-i}) - t_i^o(\theta_i^{k+q},\theta'_{-i}) + t_i^o(\theta_i^k,\theta'_{-i}) = \quad (7.29) \\
-\sum_{l=k}^{k+q-1}\sum_{z=p}^{p+m-1}\min_{\theta\in\Theta}[V_i(x(\theta_i^{l+1},\theta_{-i}^{z+1}),\theta) - V_i(x(\theta_i^l,\theta_{-i}^{z+1}),\theta) \\
-V_i(x(\theta_i^{l+1},\theta_{-i}^z),\theta) + V_i(x(\theta_i^l,\theta_{-i}^z),\theta)] = \quad (7.30) \\
-\sum_{l=k}^{k+q-1}\sum_{w=1}^{s-1}\min_{\theta\in\Theta}[V_i(x(\theta_i^{l+1},\hat\theta_{-i}^{w+1}),\theta) - V_i(x(\theta_i^l,\theta_{-i}^{w+1}),\theta) \\
-V_i(x(\theta_i^{l+1},\theta_{-i}^w),\theta) + V_i(x(\theta_i^l,\theta_{-i}^w),\theta)]. \quad (7.31)
\end{aligned}
$$

Since the supermodularity of V_i is equal to

$$
\begin{aligned}
\sum_{l=k}^{k+q-1}\sum_{w=1}^{s-1}[V_i(x(\theta_i^{l+1},\theta_{-i}^{w+1}),\theta) - V_i(x(\theta_i^l,\theta_{-i}^{w+1}),\theta) \\
- V_i(x(\theta_i^{l+1},\theta_{-i}^w),\theta) + V_i(x(\theta_i^l,\theta_{-i}^w),\theta)] \quad (7.32)
\end{aligned}
$$

and all of the summands involve one-step supermodularities, it holds that

$$
\begin{aligned}
V_i(x((\theta_i^{k+q},\theta''_{-i}),\theta) - V_i(x(\theta_i^k,\theta''_{-i}),\theta) - V_i(x(\theta_i^{k+q},\theta'_{-i}),\theta) + V_i(x(\theta_i^k,\theta'_{-i}),\theta)] \\
+ [t_i^o(\theta_i^{k+q},\theta''_{-i}) - t_i^o(\theta_i^k,\theta''_{-i}) - t_i^o(\theta_i^{k+q},\theta'_{-i}) + t_i^o(\theta_i^k,\theta'_{-i})] \geq 0 \quad (7.33)
\end{aligned}
$$

and the multiple-step supermodularity of t_i^o is the smallest possible, so that all one-steps are minimally supermodular.

Case 2. If $s > m$, it means that $s - m$ immediate successors $\tilde{\theta}''_{-i}$ under $\geq_{-i}$ are in the same category as their immediate predecessors $\tilde{\theta}'_{-i}$ and are disregarded in the sum (7.30). However, note that for all of these successors, it holds that:

$$V_i(x((\theta_i^{k+1}, \tilde{\theta}''_{-i}), \theta) - V_i(x(\theta_i^k, \tilde{\theta}''_{-i}), \theta) - V_i(x(\theta_i^{k+1}, \tilde{\theta}'_{-i}), \theta) + V_i(x(\theta_i^k, \tilde{\theta}'_{-i}), \theta)] = 0 \tag{7.34}$$

and hence equality between (7.30) and (7.31) prevails. The rest of the argument for this case follows that for case 1.

Steps 1 and 2 prove that transfers t_i^o minimally supermodular implement the decision rule x under the chosen profile of total orders $\{\geq_i^1\}_i$.

Proof of Proposition 3 By way of contradiction, suppose that profile $\theta^*(\cdot) \geq \theta^T(\cdot)$ is an equilibrium so that player i's best response to $\theta^*_{-i}(\cdot)$ is $\theta_i^*(\cdot)$. Thus, for all i, θ_i, and $\hat{\theta}_i$ such that $\theta_i^*(\theta_i) >_i^1 \hat{\theta}_i \geq_i^1 \theta_i$, the following must hold

$$E_{\theta_{-i}}[\Delta u_i(\theta^*_{-i}(\theta_{-i}); \theta)] \equiv E_{\theta_{-i}}[u_i(\theta_i^*(\theta_i), \theta^*_{-i}(\theta_{-i}); \theta)] - E_{\theta_{-i}}[u_i(\hat{\theta}_i, \theta^*_{-i}(\theta_{-i}); \theta)] \geq 0. \tag{7.35}$$

We will show that this condition is not satisfied if the inequality in the theorem holds, i.e. there must be a player for whom a deception closer to the truthful strategy is strictly better than $\theta_i^*(\cdot)$. For simplicity, define[10]

$$E_{\theta_{-i}}[\Delta V_i(\theta_{-i}; \hat{\theta}_i, \theta_{-i})] \equiv E_{\theta_{-i}}[V_i(x(\theta_i^*(\theta_i), \theta_{-i}); \hat{\theta}_i, \theta_{-i})] - E_{\theta_{-i}}[V_i(x(\hat{\theta}_i, \theta_{-i}); \hat{\theta}_i, \theta_{-i})]. \tag{7.36}$$

It follows from (5.1) and the definition of $\bar{K}_i(\theta_i)$ that for each i and θ_i:

$$E_{\theta_{-i}}[\Delta u_i(\theta^*_{-i}(\theta_{-i}); \theta)] \leq E_{\theta_{-i}}[\Delta u_i(\theta_{-i}; \theta)] + d_i(\theta_i^*(\theta_i), \hat{\theta}_i)\bar{K}_i(\theta_i) \sum_{j \neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j), \theta_j)]. \tag{7.37}$$

Since the social choice function (x, t) is implementable, the transfers $\{t_i\}$ induce truthful revelation. Therefore, it must be that for all i and θ_i the incentive compatibility constraint is satisfied, that is:

$$\begin{aligned} E_{\theta_{-i}}[V_i(x(\hat{\theta}_i, \theta_{-i}); \hat{\theta}_i, \theta_{-i})] - E_{\theta_{-i}}[V_i(x(\theta_i^*(\theta_i), \theta_{-i}); \hat{\theta}_i, \theta_{-i})] \geq \\ E_{\theta_{-i}}[t_i(\theta_i^*(\theta_i), \theta_{-i})] - E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]. \end{aligned} \tag{7.38}$$

[10] The notation we used in Equation (5.5) becomes cumbersome in this proof, and so we replace $V_i(\hat{\theta}_i \triangleright \theta_i^*(\theta_i), \theta_{-i}; \hat{\theta}_i, \theta_{-i})$ with $\Delta V_i(\theta_{-i}; \hat{\theta}_i, \theta_{-i})$.

Thus, we obtain

$$\begin{aligned}
E_{\theta_{-i}}[\Delta u_i(\theta_{-i};\theta)] &= E_{\theta_{-i}}[V_i(x(\theta_i^*(\theta_i),\theta_{-i});\theta_i,\theta_{-i})] - E_{\theta_{-i}}[V_i(x(\hat{\theta}_i,\theta_{-i});\theta_i,\theta_{-i})] \\
&\quad + E_{\theta_{-i}}[t_i(\theta_i^*(\theta_i),\theta_{-i})] - E_{\theta_{-i}}[t_i(\hat{\theta}_i,\theta_{-i})] \\
&\leq E_{\theta_{-i}}[\Delta V_i(\theta_{-i};\theta_i,\theta_{-i})] - E_{\theta_{-i}}[\Delta V_i(\theta_{-i};\hat{\theta}_i,\theta_{-i})] \\
&\leq -E_{\theta_{-i}}[\gamma_i(\theta_{-i})]d_i(\theta_i^*(\theta_i),\hat{\theta}_i)d_i(\hat{\theta}_i,\theta_i). \qquad (7.39)
\end{aligned}$$

where the first inequality is derived after substituting in the LHS of (7.38) and the second inequality follows from (5.2). Combining (7.37) and (7.39), we arrive at

$$\frac{E_{\theta_{-i}}[\Delta u_i(\theta^*_{-i}(\theta_{-i});\theta)]}{d_i(\theta_i^*(\theta_i),\hat{\theta}_i)} \leq \bar{K}_i(\theta_i)\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})]d_i(\hat{\theta}_i,\theta_i). \quad (7.40)$$

If there exist i, θ_i, and $\hat{\theta}_i \in [\theta_i, \theta_i^*(\theta_i))$ such that

$$\bar{K}_i(\theta_i)\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})]d_i(\hat{\theta}_i,\theta_i) < 0 \quad (7.41)$$

then by (7.40) $E_{\theta_{-i}}[\Delta u_i(\theta^*_{-i}(\theta_{-i});\theta)] < 0$, which contradicts (7.35). Therefore, $\theta^*(\cdot)$ is not a Bayesian equilibrium.

The same reasoning applies when $\theta^*(\cdot) \leq \theta^T(\cdot)$: if the condition of the theorem holds, $\theta^*(\cdot)$ cannot be a Bayesian equilibrium. Q.E.D

Proof of Proposition 4 Take any profile $\theta^*(\cdot) \geq \theta^T(\cdot)$ such that $E_{\theta_i}[d_i(\theta_i^*(\theta_i),\theta_i)] \geq E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)]$ for all j implies $(\theta_i, \theta_i^*(\theta_i)) \neq \emptyset$ for some θ_i. Since the scf is supermodular implementable, there exist a smallest and a largest equilibrium. By way of contradiction, suppose that $\theta^*(\cdot)$ is the largest equilibrium. Then, for all i and θ_i, the following must hold:

$$E_{\theta_{-i}}[\Delta u_i(\theta^*_{-i}(\theta_{-i});\theta)] = E_{\theta_{-i}}[u_i(\theta_i^*(\theta_i),\theta^*_{-i}(\theta_{-i});\theta)] - E_{\theta_{-i}}[u_i(\hat{\theta}_i(\theta_i),\theta^*_{-i}(\theta_{-i});\theta)] \geq 0 \quad (7.42)$$

for all deceptions $\hat{\theta}_i(\cdot) \in [\theta_i^T(\cdot), \theta_i^*(\cdot)]$. From (7.40), we know that

$$\frac{E_{\theta_{-i}}[\Delta u_i(\theta^*_{-i}(\theta_{-i});\theta)]}{d_i(\theta_i^*(\theta_i),\hat{\theta}_i(\theta_i))} \leq \bar{K}_i(\theta_i)\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})]d_i(\hat{\theta}_i(\theta_i),\theta_i) \quad (7.43)$$

for all i. Since $\theta^*(\cdot)$ is an equilibrium, the rhs of (7.43) must be nonnegative for all i, θ_i and $\hat{\theta}_i(\cdot)$. Thus, if we fix any deception $\hat{\theta}_i(\cdot)$, the expected value (over θ_i) of the rhs of (7.43) must be nonnegative for all i. We will show that there is an agent i and a strategy $\hat{\theta}_i(\cdot)$ for which this is not true, which will lead to a contradiction. Pick agent i such that $E_{\theta_i}[d(\theta_i^*(\theta_i), \theta_i)] \geq E_{\theta_j}[d(\theta_j^*(\theta_j), \theta_j)]$ for all j. Since $\theta^*(\cdot)$ is different from truthtelling, $E_{\theta_i}[d(\theta_i^*(\theta_i), \theta_i)] > 0$. Let us show that agent i, who is the agent lying the most on average, has an incentive to deviate from $\theta_i^*(\cdot)$. Note

$$\frac{\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j), \theta_j]}{E_{\theta_i}[d_i(\theta_i^*(\theta_i), \theta_i)]} \leq (n-1) < \frac{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]}{E_{\theta_i}[\bar{K}_i(\theta_i)]}, \tag{7.44}$$

hence

$$E_{\theta_i}[\bar{K}_i(\theta_i)] \sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j), \theta_j] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})] E_{\theta_i}[d_i(\theta_i^*(\theta_i), \theta_i)] < 0. \tag{7.45}$$

By the definition of a metric,

$$E_{\theta_i}[d_i(\theta_i^*(\theta_i), \theta_i)] \leq E_{\theta_i}[d_i(\theta_i^*(\theta_i), \hat{\theta}_i(\theta_i))] + E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i), \theta_i)]$$

and

$$E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i), \theta_i)] \leq E_{\theta_i}[d_i(\theta_i^*(\theta_i), \hat{\theta}_i(\theta_i))] + E_{\theta_i}[d_i(\theta_i^*(\theta_i), \theta_i)]$$

hence

$$|E_{\theta_i}[d_i(\theta_i^*(\theta_i), \theta_i)] - E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i), \theta_i)]| \leq E_{\theta_i}[d_i(\theta_i^*(\theta_i), \hat{\theta}_i(\theta_i))].$$

Therefore, there is $\underline{\varepsilon}$ small enough such that if $E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i), \theta_i^*(\theta_i))] \leq \underline{\varepsilon}$, then by (7.45)

$$E_{\theta_i}[\bar{K}_i(\theta_i)] \sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j), \theta_j] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})] E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i), \theta_i)] < 0. \tag{7.46}$$

Since there are only finitely many profiles $\theta^*(\cdot)$ (outside the neighborhood of truthtelling; see Definition 9) – because types are finite – we can choose a uniform $\underline{\varepsilon}$ such that (7.46) holds, starting from any such profile. By assumption, $\theta^*(\cdot)$ was defined to be far enough from truthtelling, i.e. $(\theta_i, \theta_i^*(\theta_i)) \neq \emptyset$ for some θ_i. Consequently, it follows from $\varepsilon(\Theta) < \underline{\varepsilon}$ that a deception $\hat{\theta}_i(\cdot) \neq \theta_i^*(\cdot)$ can be chosen such that $E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i), \theta_i^*(\theta_i))] < \underline{\varepsilon}$. In this case, (7.46) must hold, and thus i has an incentive to deviate from $\theta_i^*(\cdot)$. Profile $\theta^*(\cdot)$ is not an equilibrium. An analogous argument applies to the case when $\theta^*(\cdot) \leq \theta^T(\cdot)$. Hence, $\theta^T(\cdot)$ is essentially unique. Q.E.D

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