BELIEFS AND RATIONALIZABILITY IN GAMES WITH COMPLEMENTARITIES

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ABSTRACT. We propose two characteristics of beliefs and study their role in shaping the set of rationalizable strategy profiles in games with incomplete information. The first characteristic, *type-sensitivity*, is related to how informative a player thinks his type is. The second characteristic, *optimism*, is related to how "favorable" a player expects the outcome of the game to be. The paper has two main results: the first result provides an upper bound on the *size* of the set of rationalizable strategy profiles; the second gives a lower bound on the change of *location* of this set. These bounds are explicit expressions that involve type-sensitivity, optimism, and payoff characteristics. Our results generalize and clarify the well-known uniqueness result of global games (Carlsson and van Damme (1993)). They also imply new uniqueness results and allow us to study rationalizability in new environments. We provide applications to supermodular mechanism design (Mathevet (2010)) and information processing errors.

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1. Introduction

In all social or economic interactions, agents' beliefs contribute to shaping the set of outcomes. In game-theoretical models, the richness of outcomes is captured by the set of rationalizable strategy profiles. The global game literature (e.g., Carlsson and van Damme (1993), Frankel et al. (2003), and Morris and Shin (2003)) suggests a perturbation of complete information that delivers a unique rationalizable equilibrium. This perturbation gives players' beliefs the right properties to obtain uniqueness. What are these properties? How do they act with the payoffs to determine the rationalizable outcome? The standard global game method does not cover games with heterogeneous beliefs (non-common prior type spaces), games played by individuals with updating biases, and Bayesian mechanism design. In these cases, our understanding of rationalizability requires an answer to the above questions.

In this paper, we study properties of type spaces that explain the size and the location of the set of rationalizable strategy profiles, where rationalizability corresponds to the concept of interim correlated rationalizability by Dekel et al. (2007). Type spaces provide the framework to model incomplete information. In our formulation, there is a state of nature, and each player has type-dependent beliefs about the state of nature and others' types. We study properties of type spaces in games with complementarities in which players only care about an aggregate of their opponents' actions, such as their average action. We also assume that there exist dominance regions, that is, "tail regions" of the state space for which the extremal actions are strictly dominant. The model applies to many classic problems such as investment games, currency crisis, search models, etc.

Our main contribution comes from the fact that the analysis does not require to specify the origin of the beliefs. As in Van Zandt and Vives (2007), our formulation of the Bayesian game is interim. Since our main conditions are defined on the interim beliefs, they are compatible with general belief formation, including heterogenous beliefs and information processing errors.

The first characteristic that we study is type-sensitivity of the beliefs. This notion has two dimensions, one for the beliefs about the state and one for the beliefs about others' types. The first dimension is the answer to the question: when a player's type increases, by how much

does he think the state will increase on average? This question gives information about how informative the player thinks his type is. The second dimension applies to the beliefs about others' types. We want to know if a player believes that other players' types increase more than his when his own type increases. To find out, we ask the following question: assuming that players other than i decrease their strategies while i's type increases, by how much will the aggregate decrease on average according to i? The answer is the second dimension of type-sensitivity. Since the player's type increases, he believes others' types will increase as well. As a result, he may believe that his opponents will play larger actions although their strategies decrease, and so a larger aggregate may occur.

The second characteristic is optimism of players' beliefs. This characteristic has two dimensions and measures how favorable a player expects the outcome to be. By convention, the outcome is more favorable if the aggregate and the state are larger. Thus, a player becomes more optimistic if, at any type, he now believes larger states and larger aggregates are more likely than before. In technical terms, an increase of optimism is represented by a first-order stochastic dominance shift of the beliefs for each type.

Before stating our two main results, recall that there exist a largest and a smallest equilibrium in supermodular games. The distance between them gives the size of the set of rationalizable strategy profiles (Milgrom and Roberts (1990)).

The first result provides an explicit upper bound on the size of the set of rationalizable strategy profiles. The second result provides an explicit lower bound on the movement of the rationalizable strategy profiles after a change of optimism. Both bounds are explicit and simple expressions involving type-sensitivity, optimism, and characteristics of the payoffs. These expressions are easy to compute in comparison to applying iterative dominance and computing the rationalizable outcomes directly.

Our results provide tools to study rationalizability in general environments.

On the one hand, the results imply new uniqueness results and promote a better understanding of global games. The global game method suggests a specific perturbation of complete information that delivers a unique rationalizable equilibrium. But many scenarios do not fit into the global game description: games with heterogeneous beliefs (non-common

prior type spaces), information processing errors in games, and Bayesian mechanism design. To study these cases, it is important to understand the properties of type spaces inherited from the global game perturbation. Type-sensitivity and optimism are such properties. The upper bound provided by the first result subsumes the global game uniqueness result and shows that uniqueness holds more generally than in global games. We illustrate this in Section 2 in an investment game. The bound also shows *explicitly* that the global game information structure dampens the complementarities to the point where a unique equilibrium survives. This formalizes arguments presented by Vives (2004) and Mathevet (2007).

On the other hand, the results allow us to study equilibrium multiplicity. While the literature has focused on uniqueness, it is important to understand and quantify equilibrium multiplicity. In supermodular mechanism design, for example, knowing the size of the equilibrium set allows us to compute the welfare loss that may be caused by bounded rationality (Mathevet (2010)). Our results show that larger type-sensitivity is conducive to tighter equilibrium sets. Furthermore, certain characteristics of equilibrium multiplicity are interesting. For example, as we move from one equilibrium to another, some players may change their equilibrium strategy more dramatically than others. It seems natural to say that a player whose equilibrium strategy varies less across equilibria is more influential in the game than one whose equilibrium strategy is more responsive. In Section 6.3, we identify the more influential players as those having higher type-sensitivity.

We apply our results to supermodular mechanism design (Mathevet (2010)) and information processing errors. The idea behind supermodular mechanisms is to design mechanisms that induce games with strategic complementarities, because they are robust to certain forms of bounded rationality (Milgrom and Roberts (1990)). Although strategic complementarities are helpful, excessive complementarities may produce new equilibria and disrupt learning.[1] This justifies the concept of an optimally supermodular mechanism, one that gives the smallest equilibrium set among all supermodular mechanisms (Mathevet (2010, Theorem 3)). But what is the size of the smallest equilibrium set? Our first result can provide an answer and

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[1]This is because Mathevet (2010) studies weak implementation and truthtelling is the only equilibrium known to be desirable.

help the designer choose the right parameter values in the mechanism. In Section 6, we also study games played by players with updating biases. Information processing errors can often be interpreted as heterogeneity in the players' priors (Brandenburger et al. (1992)), hence type-sensitivity and optimism are useful. We argue that the underreaction bias favors multiplicity, while the overreaction bias favors uniqueness.

The importance of understanding rationalizability beyond global games is emphasized by Morris and Shin (2009). They characterize the hierarchies of beliefs that imply dominance-solvability in binary-action games with incomplete information. Our paper formulates alternative conditions in games with finitely many actions. We will discuss the relationship between type-sensitivity and their notion of decreasing rank beliefs. Izmalkov and Yildiz (2010) is another paper close to ours. The authors introduce optimism into the study of global games. Our second result is a generalization of their results in the partnership game. Other papers, such as Weinstein and Yildiz (2007) and Oyama and Tercieux (2011), study rationalizability in general environments, but their objective is different from ours. Weinstein and Yildiz (2007) show that for any rationalizable action of any type, the beliefs of this type can be perturbed in a way that this action is uniquely rationalizable. As a result, the beliefs may satisfy the conditions for dominance-solvability — high type-sensitivity for example — yet the unique equilibrium may vary with other properties of the beliefs — optimism for example.[2]

The paper is organized as follows. The next section gives a motivating example. Section 3 presents the model and the assumptions. Sections 4 and 5 contain the main definitions and the results. In Section 6, we provide two applications. The last section concludes.

2. An Investment Game

Consider a standard investment game. Two players are deciding whether to invest (1) or not (0). The profits are given by the following matrix where $\theta \in \mathbb{R}$ is the fundamental of the economy:

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[2]The analyst may know that there is a unique equilibrium, but without further knowledge of players' beliefs, such as their optimism level, she may be unable to pin it down, which is a form of multiplicity.

	1	0
1	θ, θ	$\theta - 1, 0$
0	$0, \theta - 1$	$0, 0$

Suppose that every player i has a type or signal t_i, and given this type, he formulates beliefs about fundamental θ and j's type t_j. Several versions of that scenario are possible:

(i) Global games. State θ is drawn from a common prior. Each investor receives a signal $t_i = \theta + \nu \epsilon_i$ where $\nu > 0$ and ϵ_i is a random variable. The analyst studies the case $\nu \to 0$ where signals become infinitely precise.

(ii) Non-common priors. Assume i's beliefs about θ given t_i are a normal distribution with mean $\frac{4t_i}{5}$ and variance σ^2. Conditionally on θ, assume i's beliefs about t_j assigns probability 1 to $t_j = \frac{3\theta}{2}$. These beliefs do not come from a common prior type space: each player i believes that j's signal is a perfect predictor of the state, while each j believes his own signal to be a noisy signal of the state. Another way of obtaining heterogeneous beliefs, proposed by Izmalkov and Yildiz (2010), is to start with the global game formulation but assume that each player has his own subjective beliefs about (ϵ_1, ϵ_2).

(iii) Subjective signaling functions. Suppose θ is drawn from a uniform prior, but players think that they obtain their private information from different channels. Player i considers that t_i is a realization of $\alpha_i^i \theta + \nu \epsilon$ and t_j is a realization of $\alpha_i^j \theta + \nu \epsilon$ where $\alpha_i^i < \alpha_i^j$. Thus, players use different signaling functions when constructing their interim beliefs. In this example, each player considers that his opponent's signal is more responsive to a shock of θ than his own signal. For some parameter values, when i's signal increases, i believes j's signal increases more. This scenario is similar in spirit to Izmalkov and Yildiz (2010) (players look at each others' signaling functions differently) and produces heterogeneous beliefs.

(iv) Non-vanishing noise. Consider the global game setup with $t_i = \theta + \nu_i \epsilon$ but let ν_i be fixed, strictly positive, and different across players (see Section 6.3).

(v) Updating Biases. Consider the global game setup with players who have updating biases. For example, players may overreact and amplify the information contained in their

signal (Section 6.2). Information processing errors can also be interpreted as heterogeneous beliefs (Brandenburger et al. (1992)).

The beliefs generated by scenarios (ii), (iii), and (v) cannot be the product of a global game formulation. Likewise, the analysis of scenarios (iv) and (v) requires new concepts.

Our main concept is *type-sensitivity*. This concept has two dimensions. Let $T_i = \mathbb{R}$ be i's type set. The first dimension is the answer to the question: if i's type increases by $v > 0$, by how much does he believe the state will increase on average? In (ii), the answer is $4v/5$.

The second dimension concerns i's beliefs about t_j. We want to know by how much j's type increases after t_i increases (according to i). Suppose an event E occurs if $\{t_j > s_j\}$ and i's type is t_i, or it occurs if $\{t_j > s_j + v\}$ and i's type is $t_i + v$. In which case does i believe E is more likely? If i believes j's type increases at least as much as his, then E is more likely in the second scenario. If it is the case and if the first dimension of type-sensitivity is strictly positive, then we refer to this as *highly type-sensitive* beliefs. For instance, beliefs are highly type-sensitive in (ii) and (iii). In the global game specification, E is more likely in the first case, but the difference in probability between the two vanishes as $\nu \to 0$.

If beliefs are highly type-sensitive, then there is a unique equilibrium. For the sake of the argument, consider the symmetric case. Let $\mu(\theta|t_i)$ be the cdf representing i's beliefs about θ given t_i. Let $\mu(\{t_j > s_j\}|\theta, t_i)$ represent i's beliefs that j's type exceeds s_j given θ and t_i. Assume that $\mu(\cdot|t_i)$ and $\mu(\cdot|\theta, t_i)$ are continuous and monotone in t_i (and θ). Monotonicity means that an increase in the state and/or own type leads to a first-order stochastic dominance shift of the belief. Under these conditions, the result by Van Zandt and Vives (2007) implies that there exist a largest and a smallest equilibrium in this investment game, and each player's (extremal-)equilibrium strategy is monotone in his type. We will show that there is a unique monotone equilibrium, and hence a unique overall equilibrium. A monotone strategy for i is characterized by a cutoff s_i. Player i invests if and only if his type is above s_i, where s_i is the type at which i is indifferent between investing and not investing:

$$\int_{\theta \in \mathbb{R}} (\theta - 1 + \mu(\{t_j > s_j\}|\theta, s_i))d\mu(\theta|s_i) = 0. \tag{2.1}$$

We prove uniqueness by way of contradiction. Suppose, instead, that the largest and the smallest equilibria are different. By symmetry, these equilibria are characterized by cutoffs $\bar{s}$ and $\underline{s}$ ($\underline{s} < \bar{s}$). By monotonicity of the beliefs, a larger type leads i to expect a strictly larger state. Define $\sigma_i^1 > 0$ to be the minimal amount by which i expects the state to increase when his type goes from $t_i = \bar{s}$ to $t_i = \underline{s}$.[3] High type-sensitivity says that i expects j to invest at least as often under strategy $\bar{s}$ given $t_i = \bar{s}$ as under strategy $\underline{s}$ given $t_i = \underline{s}$. Formally, $\mu(\{t_j > \bar{s}\}|\theta + \sigma_i^1, \bar{s}) \geq \mu(\{t_j > \underline{s}\}|\theta, \underline{s})$. If (2.1) holds at $(s_1, s_2) = (\underline{s}, \underline{s})$, which is the case by definition of an equilibrium, then

$$\int_{\theta \in \mathbb{R}} (\theta + \sigma_i^1 - 1 + \mu(\{t_j > \bar{s}\}|\theta + \sigma_i^1, \bar{s}))d\mu(\theta|\underline{s}) > 0. \tag{2.2}$$

The lhs of (2.2) is weakly smaller than the lhs of (2.1) evaluated at $(s_1, s_2) = (\bar{s}, \bar{s})$ (this is because increasing t_i from $\underline{s}$ to $\bar{s}$ increases θ by *at least* σ_i^1). Thus (2.1) does not hold at $(s_1, s_2) = (\bar{s}, \bar{s})$, which contradicts that $\bar{s}$ is an equilibrium.

The intuition is that when we move from $\underline{s}$ to $\bar{s}$, the players do not actually expect a smaller action from their opponent when their beliefs are highly type-sensitive. However, they do expect the state to increase, and thus, they cannot be indifferent between both actions. This violates requirement (2.1) for an equilibrium cutoff type. This argument is used by Frankel et al. (2003, p.8-11) to obtain uniqueness in global games, showing that type-sensitivity is the driving force behind uniqueness.

3. The Model

We study games with incomplete information. The set and the number of players are $N < \infty$. Player i's action set is $A_i = \{a_{i,1}, \ldots, a_{i,M_i}\} \subset \mathbb{R}$ where actions are indexed in increasing order. Let $A_{-i} = \prod_{i \neq i} A_j$ be the set of action profiles for players other than i. Let $\theta \in \Theta = \mathbb{R}$ be the state of nature.

3.1. **The Payoffs.** Each player i only cares about an aggregate Γ_i of his opponents' actions. This aggregate is an increasing function that maps action profiles and states from $A_{-i} \times \mathbb{R}$ *onto* a linearly ordered set $\mathcal{G}_i$. For example, a player may care about the average of his opponents'

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[3]We will call this condition strict first-order stochastic dominance.

actions or the proportion of them playing some action.[4] Moreover, assume that all payoff functions have common values, $u_i(a_i, \Gamma_i(a_{-i}, \theta), \theta)$, or private values, $u_i(a_i, \Gamma_i(a_{-i}), t_i)$, but no mixture of the two. That is, a player's utility does not depend both on the state and on his type.

The Assumptions. Let X and T be two ordered sets. A function $f : X \times T \to \mathbb{R}$ has increasing differences in (x, t) if for all $x' > x$ and $t' > t$, $f(x', t') - f(x, t') \geq f(x', t) - f(x, t)$. The function has strictly increasing differences if the previous inequality holds strictly. The assumptions are given in the common value formulation, but the same ones — replacing the state by the type — must hold under private values:

(A1) For each i and θ, u_i has increasing differences in (a_i, a_{-i}).

(A2) There are states $\overline{\theta}$ and $\underline{\theta}$ such that for every i, if $\theta > \overline{\theta}$ then the largest action is strictly dominant, and if $\theta < \underline{\theta}$ then the smallest action is strictly dominant.

(A3) For each i and a_{-i}, u_i has strictly increasing differences in (a_i, θ).

(A4) For each i and a, u_i is bounded on all compact sets of θ.

The first assumption introduces strategic complementarities: a player wants to increase his action when others do so as well. The second assumption imposes dominance regions. The third introduces state monotonicity: a player wants to increase his action when the state becomes larger. The last assumption is a technical condition.

All these assumptions are standard in the global game literature. Frankel et al. (2003) make similar assumptions on payoffs: their assumptions A1 and A3 are the same as ours; their version of A2 is weaker than ours, but our version of A4 is weaker than theirs.[5] The currency crisis model of Morris and Shin (1998), the bank run model of Morris and Shin (2000), and the model of merger waves of Toxvaerd (2008) are applications in which our assumptions A1-A4 hold. We refer the reader to Morris and Shin (2003) for further examples.

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[4]Formally, $\Gamma_i(a_{-i}, \theta) = \sum_{j \neq i} a_j$ or $\Gamma_i(a_{-i}, \theta) = (\sum_{j \neq i} 1_{a_j \geq a^*(\theta)})/(N-1)$.

[5]Frankel et al. (2003)'s A2 (for finite games) assumes that, for extreme states, extreme actions form a unique Nash equilibrium (but not necessarily in dominant strategies). Frankel et al. (2003)'s A4 assumes continuity of u_i in θ but we do not. Discontinuity is common in models with binary outcomes (currency crisis, etc).

Similar assumptions appear in Van Zandt and Vives (2007). The authors assume A1 and A3, but only require $u_i(a, \cdot) : \Theta \to \mathbb{R}$ to be measurable, and A2 is dropped. Moreover, they allow for more general type sets and action spaces.

3.2. **The Beliefs.** We use an interim formulation of a Bayesian game, based on interim beliefs, rather than a common prior (as Van Zandt and Vives (2007)). A flexible framework for modeling beliefs is that of type spaces. A type space is a collection $\mathcal{T} = (T_i, \mu_i)_{i \in N}$. Let $T_i = \mathbb{R}$ for each $i \in N$ and let $T_{-i} = \prod_{j \neq i} T_j$. For a measurable space Z, let $\Delta(Z)$ be the space of probability measures on Z. Player i's beliefs are given by

$$\mu_i : T_i \to \Delta(\Theta \times T_{-i})$$

where $\mu_i(t_i)$ is i's beliefs about the state and others' types when his type is t_i. For practical reasons, we decompose $\mu_i(t_i)$ into two beliefs: $\mu_i(\theta|t_i)$ is (the cdf of) the marginal distribution of θ and $\mu_i(\cdot|\theta, t_i)$ is the conditional measure on T_{-i} given θ. Therefore, when his type is t_i, i believes that $\theta \in \hat{\Theta}$ and $t_{-i} \in \hat{T}_{-i}$ with probability $\int_{\hat{\Theta}} \mu_i(\hat{T}_{-i}|\theta, t_i) d\mu_i(\theta|t_i)$.

Under private values, there is no state of nature, which is technically equivalent to a common values case where $\mu_i(\cdot|t_i)$ is derived from the Dirac measure.[6]

The Assumptions. Let $\geq_{st}$ stand for the first-order stochastic dominance order (see Shaked and Shanthikumar (1994, p.114) for the multivariate version).[7] Let $>_{st}$ stand for the (strict) first-order stochastic dominance order.[8] We abuse notation and apply these dominance orders to cdf and measures (with the understanding that the order actually refers to the underlying random variable or vector). We impose the following assumptions on beliefs:

(B1) For each i, if $t'_i > t_i$, then $\mu_i(\cdot|t'_i) >_{st} \mu_i(\cdot|t_i)$.

(B2) For each i, if $(t'_i, \theta') \geq (t_i, \theta)$, then $\mu_i(\cdot|\theta', t'_i) \geq_{st} \mu_i(\cdot|\theta, t_i)$.

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[6]The Dirac measure gives measure 1 to every set that contains t_i and 0 to others. It implies that all expected terms of the form $\int_{\mathbb{R}} u(\theta) d\mu_i(\theta|t_i)$ are simply equal to $u(t_i)$ for every function u.

[7]An important property is that if two random vectors X and Y in $\mathbb{R}^n$ are such that $X \geq_{st} Y$, then $u(X) \geq_{st} u(Y)$ for any increasing $u : \mathbb{R}^n \to \mathbb{R}$.

[8]Here $X >_{st} Y$ means that for every strictly increasing u, $Eu(X) > Eu(Y)$.

(B3) For each i, there is $D_i > 0$ such that $|t_i - \theta| > D_i$ implies that there exists $\bar{\epsilon} > 0$ for which $\mu_i(\theta + \epsilon|t_i) - \mu_i(\theta - \epsilon|t_i) = 0$ for all $\epsilon \in [0, \bar{\epsilon})$.

(B4) For each i, $\int_{\mathbb{R}} \mu_i(\{t_j > s_j\}_{j \neq i}|\theta, t_i) d\mu_i(\theta|t_i)$ is continuous in t_i and s_{-i}.

The first two assumptions imply monotone beliefs (i.e., $\mu_i(t_i)$ is increasing in t_i w.r.t. first-order stochastic dominance). The first one says that a player believes that larger states are more likely when his type increases. The second one says that a player believes that others' types are likely to be larger when his type and the state increase. By the third assumption, the likelihood of states that are excessively far from a player's type is null. Under private values, (B1) and (B3) hold automatically.

These assumptions are satisfied by the global-game information structure (Frankel et al. (2003)) and by most global-game applications (see Morris and Shin (2003)).[9] Van Zandt and Vives (2007) also make related assumptions. They assume that $\mu_i(t_i)$ is increasing in t_i w.r.t. $\geq_{st}$, which is slightly weaker than B1 and B2, but they do not assume B3 and have a measurable version of B4.

In view of B1-B4, belief formation is rather general in our model. Players may or may not share the same prior distribution; they may or may not exhibit biases in their updating process (see Section 6.2); etc.

3.3. **Strategies and Aggregate Distribution.** A strategy for player i is a function $s_i : T_i \rightarrow A_i$. Given our assumptions, the result by Van Zandt and Vives (2007) shows that there exist a largest and a smallest equilibrium in our model, both in monotone strategies. These extremal equilibria correspond to the largest and the smallest rationalizable strategy profiles. Since we are interested in the size and the shift of the rationalizable set, we focus on monotone strategies. Note that the finiteness of the action sets implies that a monotone

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[9]In some global games, the likelihood of states that are excessively far from a player's type must be small enough but not necessarily zero (e.g., if the noise variable has unbounded support). This can be accommodated if u_i is continuous in θ. Importantly, B3 serves two purposes: it guarantees that extreme types all have the same dominant action, and together with A4 and B4, it ensures continuity of the expected utility.

strategy is a step function, represented by a vector of cutoffs $s_i = (s_{i,\ell})_{\ell=1}^{M_i-1}$ in $\mathbb{R}^{M_i-1}$. A strategy profile is $s = (s_i)_{i \in N}$.

Given that the games under study are aggregative, every player i cares about the joint distribution of θ and Γ_i. Conditional on type t_i, i constructs $\mu_i(\theta|t_i)$, and then conditional on θ and t_i, i constructs the distribution of Γ_i as a function of others' strategies s_{-i}. This derivation is relegated to Appendix A. Let $g_i(\gamma|\tau_i)$ be the probability of $\{\Gamma_i = \gamma\}$ given $\tau_i = (\theta, s_{-i}, t_i)$. Let G_i be the corresponding cdf, i.e., $G_i(\gamma|\tau_i)$ is the probability of $\{\Gamma_i < \gamma\}$ given τ_i. Define $\Gamma_i^e[G_i(\tau_i)]$ to be the average aggregate value under $G_i(\tau_i)$.

3.4. **Rationalizability.** Our solution concept corresponds to interim correlated rationalizability (Dekel et al. (2007)). Morris and Shin (2009) note that there is no difference between ex-ante and interim rationalizability in this environment due to the supermodularity assumptions. The largest (smallest) equilibrium correspond to the largest (smallest) rationalizable strategy profile of the incomplete information game.

4. Type-sensitivity and Rationalizability

This section studies the role of type-sensitivity in determining the size of the set of rationalizable strategy profiles.

Let the distance between profiles s and s' be given by the sup norm $d(s, s') = \max_i \max_\ell |s'_{i,\ell} - s_{i,\ell}|$. Convergence under d is equivalent to convergence in measure.

4.1. **Type-sensitivity.** Since a player formulates marginal beliefs about the state and conditional beliefs about others' types, type-sensitivity has two dimensions.

Definition 1. *Type-sensitivity of the marginal beliefs is given by function σ_i^1 where for each $v > 0$, $\sigma_i^1(v)$ is the supremum of all σ such that $\mu_i(\theta - \sigma|t_i) \geq \mu_i(\theta|t_i + v)$ for all θ and t_i.*

This definition describes the minimal shift in i's marginal beliefs caused by an increase in his type. Thus, $\sigma_i^1(v)$ is a *lower bound* on how much i's beliefs about the common component change (w.r.t. first-order stochastic dominance) when his type increases by v.

If the marginal beliefs belong to a location-scale family,[10] such as the normal or logistic distribution, then $\sigma_i^1(v)$ is simply the answer to the question: when player i's type increases by v, by how much will the average state increase according to i?

The second dimension of type-sensitivity concerns the conditional beliefs $\mu_i(\cdot|\theta, t_i)$. As shown in Section 2, the driving force behind uniqueness is that when an equilibrium profile decreases, certain types of player i (those that are indifferent between two actions; call them *critical types*) should not expect Γ_i to decrease. To see why, note that when players decrease their strategies, they decide to switch to a higher action at a larger type than before. *But at this larger type, player i now believes that his opponents also have larger types* by B1 and B2; so i may actually believe (at that type) that Γ_i has increased although his opponents have decreased their strategies. The critical types who believe θ and Γ_i are larger than before cannot remain indifferent between the two actions (because of A1 and A3) and hence remain critical types. But if new critical types cannot be obtained by increasing some existing critical types, then there cannot be two distinct ordered equilibria.

The key to this argument is whether i believes that other players' types increase at least as much as his when his own type increases, and if not, by how much they fall short. Consider the two-player case and suppose i's type increases by v. Let us compare i's beliefs about t_j before and after t_i increases. By Definition 1, i's beliefs about the state shift up by at least $\sigma_i^1(v)$, i.e., i expects the state to be larger by at least that amount. Therefore, we want to compare distributions $\mu_i(\cdot|\theta, t_i)$ and $\mu_i(\cdot|\theta + \sigma_i^1(v), t_i + v)$. By assumption B2, the latter distribution stochastically dominates the former. By how much? One answer, in the spirit of Definition 1, is to define $\sigma_i^2(v)$ as the supremum of all σ such that $\mu_i(\{t_j > s_j\}|\theta + \sigma_i^1(v), t_i + v) \geq \mu_i(\{t_j + \sigma > s_j\}|\theta, t_i)$ for all parameter values. This definition gives the minimal shift in i's conditional beliefs (expressed as an increase in j's type) caused by an increase in i's type. To find out whether i believes j's type increases more than his, we can compare $\sigma_i^2(v)$ to v. But we can also ask the question directly: is the event $\{t_j > s_j + v\}$ more likely under $\mu_i(\cdot|\theta + \sigma_i^1(v), t_i + v)$ than the event $\{t_j > s_j\}$ is under $\mu_i(\cdot|\theta, t_i)$? If the

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[10]Let $f(x)$ be a pdf. The family of pdfs $(1/\eta)f((x-k)/\eta)$ indexed by (k, η), $\eta > 0$, is called the location-scale family with standard pdf f. For example, $\mu_i(\cdot|t_i)$ could be the cdf of $N(t_i/2, \xi^2)$.

answer is yes, then i must believe that j's type increases at least as much as his. If s_j were a cutoff in j's strategy, that last question would be identical to asking i whether he expects a larger action from j when t_i increases by v and j decreases his strategy by v;[11] or with many players, whether i expects a larger aggregate.

Let $\boldsymbol{v} = (v, \ldots, v)$. Let $c(v)$ be the vector such that $\tau_i + c(v) = (\theta + \sigma_i^1(v), s_{-i} + \boldsymbol{v}, t_i + v)$ represents the situation in which i's type and every j's cutoff increase by v.

Definition 2. *Type-sensitivity of the conditional beliefs is given by function σ_i^2 defined as*
$$\sigma_i^2(v) = \sup_{\tau_i} \left\{ \Gamma_i^e[G_i(\tau_i) \vee G_i(\tau_i + c(v))] - \Gamma_i^e[G_i(\tau_i + c(v))] \right\} \text{ for each } v.[12]$$

This definition provides an *upper bound* on how much player i thinks the average aggregate will decrease if his opponents decrease their strategies while his type increases by the same amount. If i believes that others' types increase more than his, then $G_i(\tau_i + c(v)) \vee G_i(\tau_i) = G_i(\tau_i + c(v))$. Then, $\sigma_i^2(v) = 0$; the player considers that the average aggregate will decrease by zero, i.e., will increase weakly. In general, the definition gives the largest amount by which the average aggregate can decrease according to i.

Consider a global-game information structure where $\theta \sim N(\mu, \xi_\theta^2)$ and $t_i = \theta + \epsilon_i$ and $\epsilon_i \sim N(0, \xi_{\epsilon_i}^2)$ for each i. By standard properties, $\theta | t_i$ is normally distributed with mean $(\xi_\theta^2 t_i + \xi_{\epsilon_i}^2 \mu)/(\xi_\theta^2 + \xi_{\epsilon_i}^2)$. Therefore, $\sigma_i^1(v) = \xi_\theta^2 v/(\xi_\theta^i + \xi_{\epsilon_i}^2)$. Furthermore, the distribution $t_j | (\theta, t_i)$ is normal with mean θ. Hence, i expects t_j to increase by $\sigma_i^1(v)$ on average when his type increases by v (because θ increases by $\sigma_i^1(v)$). Therefore, in this environment, type-sensitivity is governed by $\sigma_i^1(\cdot)$ (see Section 4.3.1).

Type-sensitivity is related to the decreasing rank beliefs condition of Morris and Shin (2009). For each k, they define rank beliefs as the probability that a player assigns to there being k players whose types are lower than his. The condition requires that as a player's type increases, he believes that his rank in the population decreases. They take the example of a student whose test score increases. Is it good news or bad news, given grading is on a

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[11]Decreasing one's strategy by v means increasing all the cutoffs in one's strategy from $s_{j,\ell}$ to $s_{j,\ell} + v$, as it delays the play of larger actions.

[12]$\vee$ stands for the supremum between two distributions w.r.t. first-order stochastic dominance. The supremum of two cdfs is the pointwise minimum between them.

curve? Under decreasing rank beliefs, it is bad news, because the student believes the test was easy, hence others' scores (their types) must have increased more than his. Thus, such a player has highly type-sensitive beliefs and his $\sigma_i^2(v)$ should be small.

4.2. **The First Theorem.** Let $\Delta_m^n u_i(\gamma, \theta) = u_i(a_n, \gamma, \theta) - u_i(a_m, \gamma, \theta)$ be the difference in i's utility between actions a_n and a_m given the aggregate is γ and the state is θ.

For every γ, let $S(\gamma) = \min\{\gamma' \in \mathcal{G}_i : \gamma' > \gamma\}$ be the successor of γ in $\mathcal{G}_i$.[13] For each θ, let

$$C_i^*(\theta) = \max_{(\gamma, n, m)} \frac{\Delta_m^n u_i(S(\gamma), \theta) - \Delta_m^n u_i(\gamma, \theta)}{(a_n - a_m)(S(\gamma) - \gamma)}. \tag{4.1}$$

In differentiable environments with a continuum of actions, (4.1) becomes $C_i^*(\theta) = \max_{(a, \gamma)}$ $\partial^2 u_i(a, \gamma; \theta)/\partial a \partial \gamma$, which is an *upper bound* on the strategic complementarities between i and the other players in the complete-information payoff at θ. Equation (4.1) is an extension of this idea to non-differentiable environments.

For every x and θ in $\mathbb{R}$, define

$$M_{i,*}(x, \theta) = \min_{(\gamma, n, m)} \frac{\Delta_m^n u_i(\gamma, \theta + x) - \Delta_m^n u_i(\gamma, \theta)}{a_n - a_m}. \tag{4.2}$$

In differentiable environments with a continuum of actions, (4.2) becomes $M_{i,*}(x, \theta) = \min_{(a, \gamma)} \partial u_i(a, \gamma; \theta + x)/\partial a - \partial u_i(a, \gamma; \theta)/\partial a$, which measures the sensitivity of i's marginal utility to an increase of the state by x. By assumption A2, there are complementarities between i's action and the state. Thus, for $x \geq 0$, (4.2) is a *lower bound* on the complementarities between i's action and the state in the complete-information payoff at θ.

Abusing notation, let $M_*(x, t_i) = \int_{\mathbb{R}} M_{i,*}(x, \theta) d\mu_i(\theta|t_i)$ and $C_i^*(t_i) = \int_{\mathbb{R}} C_i^*(\theta) d\mu_i(\theta|t_i)$ be the expected values of those functions at type t_i.

The main result features function ε

$$\varepsilon(\mu, \boldsymbol{u}) = \inf\{\underline{v} > 0 : v > \underline{v} \Rightarrow M_{i,*}(\sigma_i^1(v), t_i) > \sigma_i^2(v) C_i^*(t_i) \text{ for all } t_i \in \mathbb{R} \text{ and } i\}. \tag{4.3}$$

Consider all the $\underline{v}$'s above which the main inequality in (4.3) holds uniformly for all types and all players. Then $\varepsilon(\mu, \boldsymbol{u})$ is the infimum of those $\underline{v}$'s.

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[13]If $A_i = \{0, 1, 2, \ldots, n\}$ for all i and Γ_i is the sum, then $S(\gamma) = \gamma + 1$.

Theorem 1. *In the game of incomplete information, the distance between any two profiles of rationalizable strategies is less than $\varepsilon(\mu, \boldsymbol{u})$.*

The proof is relegated to the appendix.

The theorem states that the distance between two rationalizable strategies is less than ε if for all $v > \varepsilon$ and all t_i and player i,

$$M_{i,*}(\sigma_i^1(v), t_i) > \sigma_i^2(v) C_i^*(t_i). \tag{4.4}$$

Here is an explanation. Suppose i's own type increases by v. This has two effects on i's action. First, because i's beliefs about the state shift up and because of the complementarity between her action and the state, i wants to choose a larger action. This is the *direct effect*; by definition of $M_{i,*}$ and $\sigma_i^1(\cdot)$, $M_{i,*}(\sigma_i^1(v), t_i)$ is a lower bound on this effect. Second, because i's beliefs about the other players' type shift up, and because the other players' strategies are monotone, and because of the strategic complementarities, i wants to choose a larger action. This is the *strategic effect*. On top of this, suppose i's opponents decrease their strategies by v.[14] This only affects i through the strategic effect: because of the strategic complementarities, i wants to choose a lower action. Therefore, the overall strategic effect is ambiguous, but by definition of C_i^* and σ_i^2, $\sigma_i^2(v) C_i^*(t_i)$ is an upper bound on the strategic effect. Inequality (4.4) says that the direct effect on i's action should exceed the strategic effect. When this holds uniformly, any two monotone equilibria must be within ε of each other, for otherwise one of them would not be an equilibrium: because the direct effect dominates, some types of player i would find it optimal to play a strictly larger action than the one prescribed in one of the two equilibria.

Theorem 1 has a nice interpretation. A type-sensitive player (with small σ_i^2) acts *as if* he were not affected much by the strategic complementarities (the term $\sigma_i^2(v) C_i^*(t_i)$). For him, small changes in type lead to large changes in his action, independently of others' choices. This "disconnects" the player from others. Therefore, type-sensitivity dampens the

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[14]The comparison between two ordered equilibria implies these changes: a player's critical types increase and his opponents' strategies decrease, where decreasing one's strategy means increasing all the cutoffs in one's strategy.

strategic complementarities and favors tight equilibrium sets. To the contrary, if beliefs are not sensitive to one's type, then they can easily be swayed by others' strategies. This gives bite to the strategic complementarities and favors wide equilibrium sets.

Two comparative statics lessons can be learned from the monotonicity properties of $\varepsilon(\cdot)$. First, if $M_{i,*}$ increases and C_i^* decreases uniformly for all i, then ε decreases. Therefore, state monotonicity tends to shrink the set of rationalizable strategy profiles whereas strategic complementarities tend to enlarge it. Indeed, state monotonicity disconnects a player from the others by making his action very sensitive to his own information, while strategic complementarities make players interdependent. Interestingly, strategic complementarities do not only favor multiplicity, but they may also enlarge the equilibrium set.

Second, if σ_i^1 increases and σ_i^2 decreases uniformly, then ε decreases. Therefore, type-sensitivity tends to shrink the set of rationalizable strategy profiles. This claim, described by Corollary 1, is strong because it holds across belief structures.

Definition 3. *Beliefs μ are more type-sensitive than beliefs $\hat{\mu}$ if for all $i \in N$, $\sigma_i^1(v) \geq \hat{\sigma}_i^1(v)$ and $\sigma_i^2(v) \leq \hat{\sigma}_i^2(v)$ for all $v > 0$.*

Corollary 1. *If beliefs μ are more type-sensitive than $\hat{\mu}$, then $\varepsilon(\mu, \boldsymbol{u}) \leq \varepsilon(\hat{\mu}, \boldsymbol{u})$.*

As type-sensitivity becomes very high, the strategic complementarities have no impact and uniqueness ensues. Say that beliefs μ are highly type-sensitive if $\sigma_i^1(v) > 0$ and $\sigma_i^2(v) = 0$ for all v and $i \in N$.

Corollary 2. *If beliefs are highly type-sensitive, then there is a unique equilibrium.*

Proof. If $\sigma_i^2(\cdot) = 0$ and $\sigma_i^1(\cdot) > 0$, then $\varepsilon(\mu, \boldsymbol{u}) = 0$, since $M_{i,*}(\sigma_i^1(v), t_i) > 0$ for all $v > 0$. $\square$

4.3. **Examples.**

4.3.1. *Investment Game.* Consider the game from Section 2. It is easy to compute $C_i^*(\theta) = 1$ and $M_{i,*}(x, \theta) = x$ for all i. By Theorem 1, the size of the equilibrium set is bounded by

$$\varepsilon(\mu, \boldsymbol{u}) = \inf\{\underline{v} > 0 : v > \underline{v} \Rightarrow \sigma_i^1(v) > \sigma_i^2(v) \text{ for all } i\}. \tag{4.5}$$

In a two-player game, the aggregate is the other player's action. Therefore, $\Gamma_i^e[G_i(\tau_i)] = \mathrm{Prob}(\Gamma_i = 1|\tau_i) = \mathrm{Prob}(t_j > s_{j1}|\theta)$. In cases (ii) and (iii) of Section 2, we argued that $\sigma_i(v) > 0$ and $\sigma_i^2(v) = 0$ for all v and i, which implies uniqueness by Corollary 2.[15] Consider now a global-game information structure where $\theta \sim N(1/2, \xi_\theta)$, $\nu = 1$, and ϵ_i has a truncated normal distribution with mean 0, variance $\xi_{\epsilon_i}^2$, and support $[-4\xi_{\epsilon_i}, 4\xi_{\epsilon_i}]$. Choose $\xi_\theta = .1$ and $\xi_{\epsilon_i} = .01$. The beliefs $\mu_i(\theta|t_i)$ and $\mu_i(t_j|\theta)$ are approximately (truncated) normal distributions: the first has mean $0.99t_i + .005$ and the second has mean θ and variance $\xi_{\epsilon_i}^2$. Therefore, $\Gamma_i^e[G_i(\tau_i + c(v))]$ takes the form of a normal cdf, whose tails are 0 and 1 due to the truncation. Computations give $\sigma_i^1(v) \approx .99v$ and $\sigma_i^2(v) = \max \Gamma_i^e[G_i(\tau_i + c(v))] - \Gamma_i^e[G_i(\tau_i + c(0))] \approx .4$. We conclude $\varepsilon(\mu, \boldsymbol{u}) \approx 0.4$.

4.3.2. *Global Games.* In global games, players have a common prior over θ, $t_i = \theta + \nu\epsilon_i$ is common knowledge, and $\nu \to 0$. The main result is uniqueness. As $\nu \to 0$, the signal becomes a perfect predictor, and hence $\lim_{\nu \to 0} \sigma_i^1(v) = v$ and $\lim_{\nu \to 0} \sigma_i^2(v) = 0$ for all v.[16] Corollary 2 implies uniqueness. Equation (4.3) describes how the global game information structure dampens the complementarities to the point where a unique equilibrium survives. This generalizes and formalizes arguments presented by Vives (2004) and Mathevet (2007). At last, when the prior is uniform, there is a unique equilibrium for all $\nu > 0$: since the prior provides no information, the interim beliefs are highly type-sensitive.

5. Optimism and Rationalizability

This section studies the role of optimism in locating the rationalizable outcomes. In the investment game of Section 2, equilibrium uniqueness does not say whether the equilibrium cutoff is $s = 1/2$ or $3/4$ or else. That is, Theorem 1 does not give the position of the rationalizable profiles within the whole space. This section addresses the question: when optimism changes, across two groups of players or two periods, how do the extremal rationalizable

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[15]Case (i) satisfies all our assumptions on beliefs. Case (iii) requires some standard conditions on ϵ to satisfy these assumptions.

[16]It is not trivial to show this because convergence has to be uniform in type and strategies.

strategies change? The answer allows us to compute the change of likelihood of an event, such as a currency attack or a bank run.

First, we define optimism. Then we measure its change across belief structures. Finally, we present the main result and apply it to the model of Izmalkov and Yildiz (2010).

5.1. **Optimism.** We compare two sets of players, or the same players at two different dates, whose beliefs are $\{\mu_i\}$ and $\{\mu_i'\}$. Let G_i and G_i' be the corresponding aggregate distributions.

Player i's beliefs become *more optimistic* if $\mu_i'(\cdot|t_i) \geq_{st} \mu_i(\cdot|t_i)$ and $G_i'(\cdot|\tau_i) \geq_{st} G_i(\cdot|\tau_i)$ for all t_i and τ_i, i.e., if every type of i believes larger states and larger aggregates are more likely. This definition is related to the notion of optimism of Izmalkov and Yildiz (2010) (see Section 5.4).

5.2. **Measuring Changes in Optimism.** The magnitude of the shift of the rationalizable strategy profiles depends on the magnitude of the shift in optimism.

Definition 4. *The change of optimism of the marginal beliefs, denoted by ω_i^1, is the supremum of all ω such that $\mu_i(\theta - \omega|t_i) \geq \mu_i'(\theta|t_i)$ for all θ and t_i.*

This definition describes the minimal shift in i's marginal beliefs from μ_i to μ_i'. Thus, $\omega_i^1(v)$ is a *lower bound* on how much i's beliefs about the common component increase (w.r.t. first-order stochastic dominance) between the two belief structures.

The change of optimism of the conditional beliefs is measured via the aggregate distribution. Take two aggregate distributions G and H. If H is more optimistic than G (i.e. $H \geq_{st} G$), then the difference in optimism is the difference in their expectations. If neither H nor G is more optimistic, then a worst-case analysis is used: if H is not more optimistic than G, then at least G does not dominate H more than $G \vee H$ does. Let $\chi(H, G, \tau_i)$ be equal to $G(\tau_i)$ if $H(\tau_i) \geq_{st} G(\tau_i)$, and $H(\tau_i) \vee G(\tau_i)$ otherwise.

Definition 5. *The change of optimism from G to H is $\omega_i^2 = \inf_{\tau_i} \left\{ \Gamma_i^e[H(\tau_i)] - \Gamma_i^e[\chi(H, G, \tau_i)] \right\}$.*

This definition gives a *lower bound* on how much the average aggregate increases from G to H, or an *upper bound* on how much the average aggregate decreases from G to H. Therefore, this definition provides a worst-case perspective on the change of optimism.

5.3. **The Second Theorem.** Besides optimism, there is another effect that affects the movement of the rationalizable outcomes. Suppose players become more optimistic but their optimism is "fragile." Although they are more optimistic, a slight decrease in type $(t_i - \epsilon)$ leads them to have the same outlook on θ and Γ_i as under their original beliefs (at type t_i). In this case, it is intuitive that the rationalizable outcomes should not change much. Thus, our result will have to take into account the robustness of optimism to changes in type.

Before stating the result, we introduce an alternative notion of type-sensitivity.

Definition 6. *Type-reactivity of the marginal beliefs is given by function ψ_i^1 where for each $v > 0$, $\psi_i^1(v)$ is the infimum of all ψ such that $\mu_i(\theta + \psi|t_i) \geq \mu_i(\theta|t_i - v)$ for all θ and t_i.*

When we compare two distributions such that one stochastically dominates the other, there are two ways of measuring the magnitude of the domination. We can measure the largest amount by which the dominated one can be shifted up while remaining dominated. Or we can measure the largest amount by which the dominant one can be shifted down while remaining dominant. Definition 6 uses the latter approach, and Definition 1 uses the former. Type-reactivity describes the maximal shift in i's marginal beliefs caused by a decrease in type. As such, $\psi_i^1(v)$ is an *upper bound* on how much i's beliefs about the common component shift down when his type decreases. By definition, type-reactivity always gives larger values than type-sensitivity, but the two only differ when the shape of the beliefs changes after a change in type. For location-scale families, changes in type translate the beliefs, but do not change their shape, and hence $\psi_i^1 = \sigma_i^1$.

Denote by $C_{i,*}(\theta)$ the *minimal* amount of strategic complementarities in the complete information game at θ. This concept is defined by replacing max with min in (4.1).[17] Abusing notation, let $C_{i,*}(t_i) = \int C_{i,*}(\theta)d\mu_i(\theta|t_i)$ be the expected value of the function at type t_i.

Let $b(v)$ be the vector such that $\tau_i - b(v) = (\theta - \psi_i^1(v) + \omega_i^1, s_{-i}, t_i - v)$ represents a player with an optimistic view of the state (ω_i^1) who then receives negative news v, which decreases the state by at most $\psi_i^1(v)$.

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[17] By assumption A1, $C_{i,*}(\theta) \geq 0$ for all θ.

Using Definition 5, let $\omega_i^2(v)$ be the change of optimism from G_i to H_v where $H_v(\tau_i) = G_i'(\tau_i - b(v))$ for all τ_i. Function $\omega_i^2(\cdot)$ measures the change of optimism between player i and his optimistic self after negative news v.

The main result features function δ

$$\delta(\mu, \mu', \boldsymbol{u}) = \sup \left\{ \overline{v} > 0 : v < \overline{v} \Rightarrow M_{i,*}(\omega_i^1 - \psi_i^1(v), t_i) + \min\{\omega_i^2(v) C_{i,*}(t_i), \omega_i^2(v) C_i^*(t_i)\} > 0 \right.$$

$$\left. \text{for all } t_i \in \mathbb{R} \text{ and } i \right\}. \quad (5.1)$$

Consider all the $\overline{v}$'s below which the main inequality in (5.1) holds uniformly for all types and all players. Then $\delta(\mu, \mu', \boldsymbol{u})$ is the supremum of those $\overline{v}$'s.

Theorem 2. *In the game of incomplete information, if every player $i \in N$ becomes more optimistic from μ_i to μ_i', then the largest and the smallest rationalizable strategy profiles increase by at least $\delta(\mu, \mu', \boldsymbol{u})$.*

The proof is relegated to the appendix.

The theorem states that the distance between the largest (smallest) rationalizable strategy profiles across two belief structures is more than δ, if for all $v < \delta$ and all t_i and player i,

$$M_{i,*}(\omega_i^1 - \psi_i^1(v), t_i) + \min\{\omega_i^2(v) C_{i,*}(t_i), \omega_i^2(v) C_i^*(t_i)\} > 0. \quad (5.2)$$

Here is an explanation. Suppose every i becomes more optimistic. This has two effects. First, because i's beliefs about the state shift up and because of assumption A2, i wants to choose a higher action at t_i. This is the direct effect; by definition of $M_{*,i}$ and ω_i^1, $M_*(\omega_i^1, t_i)$ is a lower bound on this effect. Second, because i's beliefs about the other players' type shift up, and because the other players' strategies are monotone, and by assumption A1, i wants to choose a higher action at t_i. This is the strategic effect; $C_{*,i}(t_i)$ is a lower bound on this effect. Therefore, if t_i was a critical type (i.e., indifferent between two actions) before i became more optimistic, then it is no longer the case, because $M_*(\omega_i^1, t_i) + C_*(t_i) > 0$ (i's incentive to increase his action is strictly positive). We want to know how bad a news it would take for t_i to become indifferent. Thus, we look at $t_i - v$ for increasingly large values

of v. This again has two effects: a direct effect, $M_*(\omega_i^1 - \psi_i^1(v), t_i)$,[18] and a strategic effect. When i receives bad news, i's optimism changes by $\omega_i^2(v)$. If $\omega_i^2(v)$ is positive, then the lower bound on the strategic effect is $\omega_i^2(v)C_{*,i}(t_i)$: despite the negative news, an optimistic i believes that the aggregate should increase by at least $\omega_i^2(v)$, hence the (positive) strategic effect will be greater than that bound. If $\omega_i^2(v)$ is negative, then the lower bound on the strategic effect is $\omega_i^2(v)C_i^*(t_i)$: player i believes that the aggregate should decrease, but by at most $\omega_i^2(v)$, hence the (negative) strategic effect will be greater than that bound. If (5.2) holds, then for any type t_i, an optimistic i can support worse news than v at t_i, because i still has an incentive to increase his action. Hence $t_i - v$ is not a critical type, and thus the extremal rationalizable profiles must move by more than v.

The theorem has several important implications. First, the more optimistic players become, the larger the increase of the rationalizable strategy profiles tends to be. This is intuitive and holds across belief structures. More interestingly, type-reactivity is involved in locating the rationalizable strategy profiles. If a player's beliefs are not type-reactive, then as he becomes more optimistic, it takes a lot of negative information to convince him that his optimism was unfounded. Thus, larger actions can be supported at much lower types and the rationalizable outcomes change a lot. This is the next corollary.

Corollary 3. *Everything else equal, if beliefs become less type-reactive and more optimistic, then the minimal amount by which the extremal rationalizable profiles must rise increases.*

Concerning the role of payoffs, state monotonicity is conducive to larger shifts in the rationalizable outcomes via M_*. The role of strategic complementarities, however, is ambiguous. On the one hand, when a player becomes more optimistic, he foresees larger aggregate values and the strategic complementarities incite him to increase his action. On the other hand, when a player receives bad news, the effect of strong complementarities is reversed. Bad news becomes worse news.

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[18]If $\omega_i^1 - \psi_i^1(v) \geq 0$, then $M_*(\omega_i^1 - \psi_i^1(v), t_i)$ is a lower bound on how much an optimistic player receiving negative news thinks the state will increase. If $\omega_i^1 - \psi_i^1(v) < 0$, then $M_*(\omega_i^1 - \psi_i^1(v), t_i)$ becomes an upper bound on how much the state will decrease according to an optimistic i receiving negative news. In both cases, it is a worst-case analysis of what happens to the state.

5.4. **Example.** Izmalkov and Yildiz (2010) study the investment game of Section 2. The state is uniformly distributed in $\mathbb{R}$ and types $t_i = \theta + \nu\epsilon_i$. But each i has his own beliefs about (ϵ_1, ϵ_2) given by $\Pr_i$. They define optimism as $\Pr_i(\epsilon_j > \epsilon_i)$, the probability with which player i believes $\{t_i > t_j\}$, which is related to his second-order beliefs. The aggregate distribution is $G_i(\tau_i) = \mathrm{Prob}_i(t_j > s_j | t_i, \theta)$, but in symmetric two-action games, the only relevant types are such that $t_i = s_j$ in equilibrium. Therefore, $G_i(\tau_i) = \Pr_i(\epsilon_j > \epsilon_i)$ and $\omega_i^2(v) \equiv \omega_i^2 = \Delta\Pr_i(\epsilon_j > \epsilon_i)$. A player becomes more optimistic according to our definition if and only if he becomes more optimistic in the sense of Izmalkov and Yildiz (2010). We already know $C^*(t_i) = C_*(t_i) = 1$ and $M_*(x, t_i) = x$ for all t_i. Given the uniform distribution, the marginal beliefs are highly type-reactive, $\psi_i^1(v) = v$. The optimism of the marginal beliefs is fixed, $\omega_i^1 = 0$. It follows from theorem 2 that

$$\delta(\mu, \mu', \boldsymbol{u}) = \sup\{\overline{v} : v < \overline{v} \Rightarrow -v + \omega_i^2 > 0, \forall t_i, i\} = \omega_i^2 = \min_i \Delta\Pr_i(\epsilon_j > \epsilon_i), \qquad (5.3)$$

which is conform to their finding. In their model, there is a unique rationalizable profile and it co-varies perfectly with optimism, as shown by (5.3).

6. Applications

6.1. **Supermodular Mechanism Design.** Consider an adaptation of Mathevet (2010)'s motivating example. A principal needs to decide the level of a public good $x \in [0, 2]$. There are two agents, 1 and 2, whose type spaces are $T_1 = T_2 = [-.3, 1.3]$. Types are independently and uniformly distributed. Preferences are quasilinear, $u_i(x, t_i) = V_i(x, t_i) + m_i$ with $V_1(x, t_1) = t_1 x - x^2$, $V_2(x, t_2) = t_2 x + \frac{x^2}{2}$, and $m_i \in \mathbb{R}$. The principal wishes to make the efficient decision $x^*(t) = t_1 + t_2$, because it maximizes the sum $V_1 + V_2$. She asks agents to report their types. Denote i's reported type by a_i. Given the reports $a = (a_1, a_2)$, the principal chooses public good level $x^*(a)$ and money transfers $m_i(a)$ for every i. If the reports are truthful, i.e., $a_i = t_i$, then the decision is efficient, since $x^*(a) = x^*(t)$. Assume $a_i \in A = \{0, \delta, 2\delta, \ldots, 1\}$ for each i where $\delta > 0$.[19] Mathevet (2010) suggests using the

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[19]There are finitely many reports so that the model fits into the framework of Section 3. Moreover, the largest and the smallest report that an agent can make are 0 and 1, whereas an agent's type lies in $[-.3, 1.3]$.

following transfers:

$$m_1(a) = \frac{13}{12} + a_1 + \frac{a_1^2}{2} + \rho_1 a_1 (a_2 - 1/2)$$

and

$$m_2(a) = -\frac{7}{6} - \frac{a_2}{2} - a_2^2 + \rho_2 a_2 (a_1 - 1/2)$$

where ρ_1 and ρ_2 have to be chosen by the designer. The utility functions $V_i(x^*(a), t_i) + m_i(a)$, $i = 1, 2$, define a private value environment. There exist values of ρ_1 and ρ_2, including resp. 2 and -1, for which the assumptions of Section 3 are satisfied. In particular, the utility functions exhibit strategic complementarities and for each i, $a_i = 1$ is strictly dominant for $t_i > \bar{t} = 1\frac{1}{4}$ and $a_i = 0$ is strictly dominant for $t_i < \underline{t} = -\frac{1}{4}$. By Theorem 1, this mechanism induces a game whose size of the equilibrium set is less than

$$\varepsilon(\mu, \boldsymbol{u}) = \inf\{\underline{v} > 0 : v > \underline{v} \Rightarrow \delta v - \sigma_2^1(v)(\rho_1 - 2) > 0 \text{ and } \delta v - \sigma_2^2(v)(1 + \rho_2) > 0\}. \quad (6.1)$$

Thus, the equilibrium set may enlarge as ρ_1 and ρ_2 increase, which underlies optimal supermodular implementation (Mathevet (2010)). Equation (6.1) also shows that the mechanism has a unique equilibrium for $\rho_1 = 2$ and $\rho_2 = -1$. For these values, the unique equilibrium is essentially truthful: if his type falls into A, a player reports it truthfully, otherwise he chooses the report closest to his type. Importantly, our conclusions hold for any $\delta > 0$.

6.2. **Updating Biases.** This section studies the strategic implications of specific updating biases. Let $p(\theta, t)$ be the joint density function of state and types. Players update their beliefs upon receiving their type, but we allow them to make mistakes when processing information (see Kahneman et al. (1982) and Epstein (2006)).

For every measurable $S = S_\theta \times S_{-i} \subset \Theta \times T_{-i}$, let

$$\mu_i^{BU}(t_i)[S] = \frac{\int_{S_\theta} \int_{S_{-i}} p(\theta, t) dt_{-i} d\theta}{\int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} p(\theta, t) dt_{-i} d\theta}$$

be the belief that $(\theta, t_{-i}) \in S$ for a Bayesian player at t_i, and let

$$\mu_i^p[S] = \int_{S_\theta} \int_{S_{-i}} \left(\int_{\mathbb{R}} p(\theta, t) dt_i \right) dt_{-i} d\theta$$

Although this implies that agents will necessarily lie when their true types are extreme, this also guarantees the existence of dominance regions and allows our framework to be useful in the present context.

be i's belief that $(\theta, t_{-i}) \in S$ prior to receiving any information. The main departure from standard treatments is that $\mu_i(t_i)$, player i's actual beliefs at t_i, might be different from $\mu_i^{BU}(t_i)$.

We will focus on specific updating biases inspired from Epstein (2006, p.420). One specification used by the author is:

$$\mu_i(t_i) = (1 - \alpha_i)\mu_i^{BU}(t_i) + \alpha_i\mu_i^p \tag{6.2}$$

where $0 \leq \alpha_i \leq 1$. In this case, μ_i is a mixture of μ_i^{BU} and μ_i^p. As Epstein (2006) writes: "Because $\mu_i^{BU}(t_i)$ embodies "the correct" combination of prior beliefs and responsiveness to data, and because μ_i^p gives no weight to data, the updating implied by (6.2) gives "too much" weight to the prior and "too little" to observation." Call this the *underreaction bias*.

Consider also the following specification

$$\mu_i(t_i) = \mu_i^{BU}(t_i + \alpha_i(t_i - t_i^*)) \tag{6.3}$$

where $\alpha_i > 0$. Player i believes at t_i what a Bayesian player would believe at $t_i + \alpha(t_i - t_i^*)$ where t_i^* is some psychological threshold. If player i receives $t_i > t_i^*$ (or $t_i < t_i^*$), i interprets his information as a better (worse) news than what it actually is. Thus, i gives "too much" weight to observation. Call this the *overreaction bias*.

Suppose that μ_i^{BU} satisfies assumptions B1-B4 and μ_i^p satisfies B4. Then (6.3) satisfies B1-B4 for all $\alpha_i > 0$ and our theorems apply. As for (6.2), if α_i is small enough for all i, then our theorems hold as well.

According to Definition 3, beliefs μ_i^{BU} are more type-sensitive than (6.2), due to the inertia of the prior. Therefore, by Corollary 1, the underreaction bias tends to favor equilibrium multiplicity and wider sets of rationalizable strategy profiles. On the other hand, beliefs (6.3) are more type-sensitive than μ_i^{BU}. Indeed, $t_i + \alpha_i(t_i - t_i^*) = (1 + \alpha_i)t_i - \alpha_i t_i^*$, and since $\alpha_i > 0$, an increase in t_i leads to a larger shift of i's beliefs. By Corollary 1, the overreaction bias promotes tighter sets of rationalizable strategy profiles and favors equilibrium uniqueness.

Information processing errors can often be interpreted as heterogeneity in the players' priors (Brandenburger et al. (1992) and Morris (1995)). To see why, consider (6.3) and suppose that α_i is large for $i = 1, 2$ and that the variance is small (i.e., μ_i puts a lot of

mass around $(1 + \alpha_i)t_i - \alpha_i t_i^*$). Then the players have very different perceptions of the joint distribution of states and types. Each player believes that his opponent's type is more extreme than his own and that it increases much faster than his own. Two players cannot commonly agree on this, unless they have different priors. The case of (6.2) is similar. If $\alpha_1 \approx 0$ and $\alpha_2 \approx 1$, then players may have different beliefs about the informativeness of each other's type. If the prior is very informative, in the sense that t_1 and t_2 are highly correlated, then 1 thinks that 2's type is a great predictor of t_1 and vice versa, while 2 considers his own type to be a poor predictor of t_1.

6.3. **Type-sensitivity and Influence.** This section investigates the relationship between type-sensitivity and a notion of influence in games: *players whose beliefs are sufficiently more type-sensitive are more influential.* This relationship is especially interesting when type-sensitivity is viewed as a form of confidence in one's information, because confidence carries influence. Behavioral economics provides many definitions of confidence, some related to the perceived precision of one's information (Odean (1999), Healy and Moore (2009)). It can be argued that type-sensitivity captures a notion of confidence in one's type, at least in variational terms. Indeed, a player may believe that his type is not an exact predictor of the state and others' types, and yet be confident that the variations in his type match the variations in the state and in others' types, which corresponds to high type-sensitivity.

Consider binary-action games. Theorem 1 says that, unless type-sensitivity is high for all players, there should be multiple equilibria. Suppose that there are many equilibria. Take any two of them, s^* and s^{**}, such that $s^* < s^{**}$. One way of measuring a player's influence is via $s_i^{**} - s_i^*$. This is the amount by which i changes his equilibrium strategy in response to changes in others' equilibrium strategies. For example, if $s_1^{**} - s_1^* < \max_{j \neq 1} s_j^{**} - s_j^*$, then any player $j \neq 1$ changes his strategy more than 1, although every $j \neq 1$ responds to a smaller change in his opponents' strategies than 1.[20] Thus, 1 is said to be more *influential*.

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[20]Player 1's opponents change their strategies more than j's opponents because $s_{-1}^{**} - s_{-1}^* > s_{-j}^{**} - s_{-j}^*$ in the product order.

Proposition 1. *For any player i, any subset $\mathcal{N} \subset N\backslash\{i\}$, and any two equilibria s^{**} and s^*, there exist $\overline{\sigma}_1(\cdot)$ and $\overline{\sigma}_2(\cdot)$ such that $\sigma_i^1(v) \geq \overline{\sigma}_1(v)$ and $\sigma_i^2(v) \leq \overline{\sigma}_2(v)$ for all $v > 0$ imply that i is more influential than any $j \in \mathcal{N}$: $s_i^{**} - s_i^* < \max_{j \in \mathcal{N}} s_j^{**} - s_j^*$.*

The proof uses arguments from the proof of Theorem 1 and is omitted.

7. Conclusion

In this paper, we have introduced type-sensitivity and generalized optimism, two notions that capture essential features of the beliefs involved in shaping the set of rationalizable strategy profiles. The main contribution of the paper is twofold. First, it does not specify the origin of the beliefs, and thus it covers a range of new scenarios. Second, it incorporates properties of the beliefs and of the payoffs into explicit expressions that deliver interesting comparative statics.

Appendix A. Aggregate Distribution

For players other than i, consider the set of types that are lower than t'_{-i} and the set of types that are larger than t'_{-i}:

$$\underline{L}(t'_{-i}) = \{t_{-i} \in T_{-i} : t_j \leq t'_j \text{ for all } j \neq i\}$$

and

$$\overline{L}(t'_{-i}) = \{t_{-i} \in T_{-i} : t_j \geq t'_j \text{ for all } j \neq i\}.$$

Let $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_N) \in \mathbb{N}^{N-1}$, and denote by $a_{-i,\boldsymbol{\ell}} \in \prod_{j \neq i} A_j$ the vector of actions in which each $j \neq i$ plays action a_{j,ℓ_j}. Define $A_{-i}(\gamma, \theta) = \{\boldsymbol{\ell} \in \mathbb{N}^{N-1} : \Gamma_i(a_{-i,\boldsymbol{\ell}}, \theta) = \gamma\}$ to be the set of action profiles that yield aggregate value γ at state θ. Recall that player j plays action a_{j,ℓ_j} if and only if his type is in $[s_{j,\ell_j-1}, s_{j,\ell_j}]$. The aggregate distribution is represented by the following probability mass function

$$g_i(\gamma|\tau_i) = \mu_i(\theta, t_i) \left[\bigcup_{\boldsymbol{\ell} \in A_{-i}(\gamma,\theta)} \left\{ \underline{L}((s_{j,\ell_j})_{j \neq i}) \bigcap \overline{L}((s_{j,\ell_j-1})_{j \neq i}) \right\} \right].$$

Let $G_i(\cdot|\tau_i)$ be the cumulative distribution function derived from g_i.

Appendix B. Proofs

The argument of the first result goes as follows:

(1) The games under consideration have strategic complements (GSC). This implies the existence of a largest and a smallest equilibrium (Milgrom and Roberts (1990) and Vives (1990)).

(2) Furthermore, the payoffs display some monotonicity between actions and states, and the beliefs display monotonicity in type. By Van Zandt and Vives (2007), (a) best-responses to monotone (in-type) strategies are monotone and (b) the extremal equilibria are in monotone strategies.

(3) We prove that the best-reply mapping, restricted to monotone strategies, is a contraction for all pairs of profiles that are distant enough. Since the extremal equilibria are in monotone strategies, they can be no further apart than this distance.

(4) Since extremal equilibria bound the set of rationalizable strategy profiles in GSC, this gives a distance between any pair of rationalizable profiles.

In view of (2), we restrict attention to monotone (in-type) strategies. Any such strategy can be represented as a finite sequence of cutoff points. Call these cutoff points *real cutoffs* as opposed to the *fictitious cutoffs* defined later. Player i's strategy is $s_i = (s_{i,\ell})_{\ell=1}^{M_i-1}$ where each $s_{i,\ell}$ is the threshold type below which i plays a_ℓ and above which he plays $a_{\ell+1}$.

Definition 7. *For each i, the fictitious cutoff between a_n and a_m, denoted $c_{n,m}$ is defined, if it exists, as the (only) type t_i such that $Eu_i(a_n, s_{-i}, t_i) - Eu_i(a_m, s_{-i}, t_i) = 0$.*

Define the expected utility as

$$Eu_i(a_i, s_{-i}, t_i) = \int_{\mathbb{R}} \sum_{\gamma \geq \underline{\gamma}} u_i(a_i, \gamma, \theta) g_i(\gamma | \theta, s_{-i}, t_i) d\mu_i(\theta | t_i). \tag{B.1}$$

Let $\Delta_m^n Eu_i(s_{-i}, t_i) = Eu_i(a_{i,n}, s_{-i}, t_i) - Eu_i(a_{i,m}, s_{-i}, t_i)$.

B.1. **Proposition 2.**

Proposition 2. *If $v > \varepsilon(\mu, \boldsymbol{u})$, then for all pairs of actions (a_n, a_m), types t_i, strategies s_{-i}, and $i \in N$ such that*

$$Eu_i(a_n, s_{-i}, t_i) - Eu_i(a_m, s_{-i}, t_i) \geq 0 \tag{B.2}$$

the following inequality holds

$$Eu_i(a_n, s_{-i} + \boldsymbol{v}, t_i + v) - Eu_i(a_m, s_{-i} + \boldsymbol{v}, t_i + v) > 0 \tag{B.3}$$

Proof. We first establish two strings of inequalities. The first string of inequalities goes as follows: for every τ_i and v, actions a_n and a_m, and player i,

$$\sum_{\gamma \geq \underline{\gamma}} \left(\Delta_m^n u_i(\gamma, \theta + \sigma_i^1(v)) - \Delta_m^n u_i(\gamma, \theta) \right) g_i(\gamma | \tau_i + c(v))$$

$$= (a_n - a_m) \sum_{\gamma \geq \underline{\gamma}} \left(\frac{\Delta_m^n u_i(\gamma, \theta + \sigma_i^1(v)) - \Delta_m^n u_i(\gamma, \theta)}{a_n - a_m} \right) g_i(\gamma | \tau_i + c(v))$$

$$\geq (a_n - a_m) M_*(\theta, \sigma_i^1(v)) \sum_{\gamma \geq \underline{\gamma}} g_i(\gamma | \tau_i + c(v))$$

$$= (a_n - a_m) M_*(\theta, \sigma_i^1(v)). \tag{B.4}$$

Let $G_i^*(\cdot|\tau_i, v)$ be the cdf of distribution $G_i(\tau_i) \vee G_i(\tau_i + c(v))$, and let $g_i^*(\cdot|\tau_i, v)$ be its probability mass function. For every τ_i, n and m, and v, notice that

$$\sum_{\gamma \geq \underline{\gamma}} \Delta_m^n u_i(\gamma, \theta)(g_i(\gamma|\tau_i + c(v)) - g_i(\gamma|\tau_i)) =$$

$$\sum_{\gamma \geq \underline{\gamma}} (G_i(S(\gamma)|\tau_i + c(v)) - G_i(S(\gamma)|\tau_i))(\Delta_m^n u_i(\gamma, \theta) - \Delta_m^n u_i(S(\gamma), \theta)). \quad \text{(B.5)}$$

The second string of inequalities goes as follows: for every τ_i, n and m, and v,

$$\sum_{\gamma \geq \underline{\gamma}} (G_i(S(\gamma)|\tau_i + c(v)) - G_i(S(\gamma)|\tau_i))(\Delta_m^n u_i(\gamma, \theta) - \Delta_m^n u_i(S(\gamma), \theta))$$

(by definition of G_i^*)

$$\geq \sum_{\gamma \geq \underline{\gamma}} (G_i(S(\gamma)|\tau_i + c(v)) - G_i^*(S(\gamma)|\tau_i, v)) \times$$

$$\left(\frac{\Delta_m^n u_i(\gamma, \theta) - \Delta_m^n u_i(S(\gamma), \theta)}{(a_n - a_m)(\gamma - S(\gamma))}\right)(a_n - a_m)(\gamma - S(\gamma))$$

(by definition of C_i^*)

$$\geq \sum_{\gamma \geq \underline{\gamma}} (G_i(S(\gamma)|\tau_i + c(v)) - G_i^*(S(\gamma)|\tau_i, v))(\gamma - S(\gamma))(a_n - a_m)C_i^*(\theta)$$

$$= C_i^*(\theta) \sum_{\gamma} \gamma(g_i(\gamma|\tau_i + c(v)) - g_i^*(\gamma|\tau_i))(a_n - a_m)$$

(by definition of σ_i^2)

$$\geq - C_i^*(\theta)\sigma_i^2(v)(a_n - a_m) \quad \text{(B.6)}$$

This last expression is a lower bound on (B.5). By definition of $\varepsilon(\cdot)$, if $v > \varepsilon(\mu, \boldsymbol{u})$, then

$$(a_n - a_m)(E_{\theta|t_i}[M_*(\theta, \sigma_i^1(v))] - \sigma_i^2(v)E_{\theta|t_i}[C_i^*(\theta)]) > 0 \quad \text{(B.7)}$$

for all types t_i, n and m. Therefore, given the inequalities described by (B.4) and (B.6), it follows from (B.7) that

$$E_{\theta|t_i}\left[\sum_{\gamma\geq\underline{\gamma}}(\Delta_m^n u_i(\gamma,\theta+\sigma_i^1(v))-\Delta_m^n u_i(\gamma,\theta))g_i(\gamma|\tau_i+c(v))\right]$$

$$+E_{\theta|t_i}\left[\sum_{\gamma\geq\underline{\gamma}}\Delta_m^n u_i(\gamma,\theta)(g_i(\gamma|\tau_i+c(v))-g_i(\gamma|\tau_i))\right]>0 \quad (B.8)$$

for all t_i, n and m, and player i. Since (B.2) holds by assumption, (B.8) implies

$$E_{\theta|t_i}\left[\sum_{\gamma\geq\underline{\gamma}}\Delta_m^n u_i(\gamma,\theta+\sigma_i^1(v))g_i(\gamma|\tau_i+c(v))\right]>0. \quad (B.9)$$

After a change of variable, we can see that (B.9) is equivalent to

$$\int_{\mathbb{R}}\sum_{\gamma\geq\underline{\gamma}}\Delta_m^n u_i(\gamma,\theta)g_i(\gamma|\theta,s_{-i}+\boldsymbol{v},t_i+v)d\mu_i(\theta-\sigma_i^1(v)|t_i)>0. \quad (B.10)$$

By definition of type-sensitivity and assumptions A1, A2, B1 and B2, the lhs of (B.10) is smaller than the lhs of (B.3), and hence (B.10) implies (B.3). This completes the proof. $\square$

B.2. **Real vs. Fictitious Cutoffs and Proposition 4.** The real cutoffs are the threshold types that separate an action from its successor. They are sufficient to represent any increasing strategy. How to recover the real cutoffs from the fictitious cutoffs?

Example 1. *(Mathevet (2007))* Consider a game with two players. Let $A_1 = A_2 = \{0,1,2\}$. There are three fictitious cutoffs, $c_{1,0}$, $c_{2,0}$ and $c_{2,1}$, but only two are needed to represent a player's best-response. Which ones? For instance, suppose strategy $(0.2, 0.8)$ is a best-response for i to some strategy s_j of player j. It consists in playing 0 for types below 0.2, 2 for types above 0.8, and 1 in between. In this case, the first real cutoff, $s_{i,1}$, that separates 0 and 1 is $0.2 = c_{1,0}$. The second real cutoff, $s_{i,2}$, that separates 1 and 2 is $0.8 = c_{2,1}$. Now, consider the following best-response $(0.4, 0.4)$ to s_j'. In this case, the player never plays 1 except possibly on a set of measure zero (when receiving exactly type 0.4). The first real cutoff, $s_{i,1}'$, that separates 0 and 1 is $0.4 = c_{2,0}'$, but the second real cutoff, $s_{i,2}'$, is also $c_{2,0}'$,

because 1 is not played. This shows that a real cutoff is not always the same fictitious cutoff, as it can take on the value of different fictitious cutoffs.

This leads to the following definition where the real cutoffs are defined recursively from the fictitious cutoffs.[21]

Definition 8. *The largest real cutoff, s_{i,M_i-1}, is the fictitious cutoff $c_{M_i,\alpha}$ such that (i) for any $t_i > c_{M_i,\alpha}$, $\Delta_k^{M_i} Eu_i(s_{-i}, t_i) > 0$ for all $k \neq M_i$, (ii) for some $\epsilon > 0$ and any $t_i \in (c_{M_i,\alpha} - \epsilon, c_{M_i,\alpha})$, $\Delta_k^\alpha Eu_i(s_{-i}, t_i) > 0$ for all $k \neq \alpha$. Suppose $s_{i,\ell} = c_{n,m}$. Then define $s_{i,\ell-1}$ as follows. If $\ell > m$, then the real cutoff $s_{i,\ell-1} = c_{n,m}$. If $\ell = m$, then $s_{i,\ell-1} = c_{m,\beta}$ such that (i) for any $t_i > c_{m,\beta}$, $\Delta_k^m Eu_i(s_{-i}, t_i) > 0$ for all $k \neq m$ and (ii) for some $\epsilon > 0$ and any $t_i \in (c_{m_i,\beta} - \epsilon, c_{m_i,\beta})$, $\Delta_k^\beta Eu_i(s_{-i}, t_i) > 0$ for all $k \neq \beta$.*

The dominance regions imply that a_{i,M_i} will be played, so the largest real cutoff is the fictitious cutoff between a_{i,M_i} and the action $a_{i,\alpha}$ played before it. All actions in between are not played, hence they receive the same real cutoff. We proceed in a downward fashion to find the action that was played before $a_{i,\alpha}$ and so on.

The next proposition shows that if an action is strictly dominated by another action for all types against *some* opposing profile, then it must be strictly dominated by that same action for all types and against *all* opposing profiles. As a result, the same set of fictitious cutoffs will exist across opposing strategy profiles.

Proposition 3. *Let $\varepsilon(\mu, \boldsymbol{u}) < \bar\theta - \underline\theta + 2\max_i D_i$.[22] For any $a_i, a_i' \in A_i$, if there is $s_{-i}' \in \mathbb{R}$ such that $Eu_i(a_i', s_{-i}', t_i) > Eu_i(a_i, s_{-i}', t_i)$ for all $t_i \in \mathbb{R}$, then $Eu_i(a_i', s_{-i}, t_i) > Eu_i(a_i, s_{-i}, t_i)$ for all s_{-i} and $t_i \in \mathbb{R}$.*

Proof. Let $\varepsilon(\mu, \boldsymbol{u}) < \bar\theta - \underline\theta + 2\max_i D_i$. Suppose first $a_i' > a_i$. If there is s_{-i}' such that $Eu_i(a_i', s_{-i}', t_i) > Eu_i(a_i, s_{-i}', t_i)$ for all t_i, then it follows from Proposition 2 that

$$Eu_i(a_i', s_{-i}' + \boldsymbol{v}, t_i + v) - Eu_i(a_i, s_{-i}' + \boldsymbol{v}, t_i + v) > 0, \tag{B.11}$$

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[21]Existence of the fictitious cutoffs poses no problem in the definition, for if a real cutoff takes on the value of a fictitious cutoff, that fictitious cutoff must exist.

[22]If $\varepsilon(\mu, \boldsymbol{u}) = \bar\theta - \underline\theta + 2D_i$, then the main result says that the size of the equilibrium set is the whole space. The result is only interesting for $\varepsilon(\mu, \boldsymbol{u}) < \bar\theta - \underline\theta + 2\max_i D_i$.

for all $v > \varepsilon(\mu, \boldsymbol{u})$ and all t_i. For any s_{-i}, choose $v > \varepsilon(\mu, \boldsymbol{u})$ such that $s'_{-i} + \boldsymbol{v} \geq s_{-i}$ (so s_{-i} is a larger strategy). Larger strategies lead to larger aggregates, hence (B.11) and the strategic complementarities imply

$$Eu_i(a'_i, s_{-i}, t_i + v) - Eu_i(a_i, s_{-i}, t_i + v) > 0$$

for all t_i. This is equivalent to saying $Eu_i(a'_i, s_{-i}, t_i) - Eu_i(a_i, s_{-i}, t_i) > 0$ for all t_i. Since s_{-i} was arbitrary, the claim is proved. Suppose now that $a'_i < a_i$. If there is s'_{-i} such that $Eu_i(a'_i, s'_{-i}, t_i) > Eu_i(a_i, s'_{-i}, t_i)$ for all t_i, then Proposition 2 implies

$$Eu_i(a'_i, s'_{-i} - \boldsymbol{v}, t_i - v) - Eu_i(a_i, s'_{-i} - \boldsymbol{v}, t_i - v) > 0 \tag{B.12}$$

for all $v > \varepsilon(\mu, \boldsymbol{u})$ and t_i. For any s_{-i}, choose $v > \varepsilon(\mu, \boldsymbol{u})$ such that $s_{-i} \geq s'_{-i} - \boldsymbol{v}$. By (B.12) and the strategic complementarities, we have

$$Eu_i(a'_i, s_{-i}, t_i - v) - Eu_i(a_i, s_{-i}, t_i - v) > 0$$

for all t_i, which is equivalent to $Eu_i(a'_i, s_{-i}, t_i) - Eu_i(a_i, s_{-i}, t_i) > 0$ for all t_i. $\qquad\square$

The next proposition is an important piece of the main theorem. If all of i's fictitious cutoffs contract in response to a variation of s_{-i}, then so do all of i's real cutoffs. That is, i's best-reponse contracts as well.

Proposition 4. *Suppose $\varepsilon(\mu, \boldsymbol{u}) < \overline{\theta} - \underline{\theta} + 2 \max_i D_i$. If, for some $v > 0$, we have $|c'_{n,m} - c_{n,m}| < v$ for all n and m for which $c'_{n,m}$ and $c_{n,m}$ exist, then $|s_{i,\ell} - s'_{i,\ell}| < v$ for all $\ell = 1, \ldots, M_i - 1$.*

Proof. The result is proved by induction. Suppose that, for some $v > 0$, $|c'_{n,m} - c_{n,m}| < v$ for all n and m for which both $c'_{n,m}$ and $c_{n,m}$ exist.

We first prove that the result holds for the largest real cutoff and then extend it to other cutoffs by induction. Let the largest real cutoff $s_{i,M_i-1} = c_{n,m}$ and $s'_{i,M_i} = c'_{k,\ell}$.

The largest action a_{i,M_i} is always played for large enough types. So the largest real cutoff always takes on the value of the fictitious cutoff between a_{i,M_i} and some other action. Suppose that $s_{i,M_i-1} = c_{M_i,w}$ and $s'_{i,M_i-1} = c'_{M_i,z}$. Proposition 3 implies that $c_{M_i,z}$ must exist. To see why, suppose $c_{M_i,z}$ did not exist. Since a_{M_i} must be played, it would mean that

a_{M_i} strictly dominates a_z for all t_i against s_{-i}. Proposition 3 would then imply that a_{M_i} strictly dominates a_z for all t_i and all opposing strategies, s'_{-i} in particular, contradicting the existence of $c'_{M_i,z}$. Therefore, $s'_{i,M_i-1} - s_{i,M_i-1} = c'_{M_i,z} - c_{M_i,w} = c'_{M_i,z} - c_{M_i,z} + c_{M_i,z} - c_{M_i,w}$. Note that $c_{M_i,z} - c_{M_i,w} \leq 0$. Indeed, $s_{i,M_i-1} = c_{M_i,w}$ implies that a_{i,M_i} is played right after $a_{i,w}$ in the best-response, hence a_{i,M_i} became preferable to $a_{i,z}$ before $c_{M_i,w}$. Since $c'_{M_i,z} - c_{M_i,z} < v$, then $s'_{i,M_i-1} - s_{i,M_i-1} < v$. The proof is similar for $s_{i,M_i-1} - s'_{i,M_i-1}$, hence $|s'_{i,M_i-1} - s_{i,M_i-1}| < v$.

For the other real cutoffs, the situation is more difficult, because the corresponding action may not be played. By induction hypothesis, suppose that $|s'_{i,\ell+1} - s_{i,\ell+1}| < v$. The objective is to show that it implies $|s'_{i,\ell} - s_{i,\ell}| < v$. There are several cases:

Case 1: Action $a_{i,\ell}$ is played both under s_i and s'_i. This case is similar to the case of the largest real cutoff, and the proof is identical.

Case 2: Action $a_{i,\ell}$ is played neither under s_i nor s'_i. By definition (8), $s_{i,\ell} = s_{i,\ell+1}$ and $s'_{i,\ell} = s'_{i,\ell+1}$. By induction hypothesis, $|s'_{i,\ell} - s_{i,\ell}| = |s'_{i,\ell+1} - s_{i,\ell+1}| < v$.

Case 3: Action $a_{i,\ell}$ is not played in s_i but it is in s'_i. Then, $s_{i,\ell} = c_{w,z}$ for some actions $a_{i,w}$ and $a_{i,z}$ such that $z < \ell < w$, and $s'_{i,\ell} = c'_{\ell,x}$ for some $a_{i,x}$. Write $s'_{i,\ell} - s_{i,\ell} = c'_{\ell,x} - c_{w,z}$.

First, we establish that both $c_{w,\ell}$ and $c'_{w,\ell}$ exist. Action $a_{i,w}$ is played (under s_i) against s_{-i} but it cannot strictly dominate $a_{i,\ell}$ for all types t_i, because if it did, then Proposition 3 would imply that it is also the case (under s'_i) against s'_{-i} (thus $a_{i,\ell}$ could not be played under s'_i, yet it is). Therefore, $c_{w,\ell}$ must exist. This implies that for all $t_i \geq c_{w,\ell}$,

$$Eu_i(a_{i,w}, s_{-i}, t_i) > Eu_i(a_{i,\ell}, s_{-i}, t_i). \tag{B.13}$$

Let $\boldsymbol{h} = (h, \ldots, h)$ where $h > \varepsilon(\mu, \boldsymbol{u})$ is large enough such that $s_{-i} + \boldsymbol{h} \geq s'_{-i}$. It follows from Proposition 2 and (B.13) that for all $t_i \geq c_{w,\ell}$,

$$Eu_i(a_{i,w}, s_{-i} + \boldsymbol{h}, t_i + h) > Eu_i(a_{i,\ell}, s_{-i} + \boldsymbol{h}, t_i + h)$$

and thus by strategic complementarities,

$$Eu_i(a_{i,w}, s'_{-i}, t_i + h) > Eu_i(a_{i,\ell}, s'_{-i}, t_i + h),$$

for all $t_i \geq c_{w,\ell}$. We know $a_{i,\ell}$ is played (under s_i') against s_{-i}', so the last inequality implies that $c_{w,\ell}'$ exists.

Second, we prove that real cutoff contracts. The following inequality must hold, $c_{w,\ell}' \geq c_{\ell,x}'$, because $a_{i,\ell}$ is played under s_i' in an open set of types above $c_{\ell,x}'$ (so it is only for types larger than $c_{\ell,x}'$ that $a_{i,w}$ can be preferred to $a_{i,\ell}$). Similarly, $c_{w,\ell} \leq c_{w,z}$, because $a_{i,w}$ is played under s_i in an open set of types above $c_{w,z}$, hence $a_{i,w}$ started to be preferred to $a_{i,\ell}$ for smaller types. As a result,

$$s_{i,\ell}' - s_{i,\ell} = c_{\ell,x}' - c_{w,z} \leq c_{w,\ell}' - c_{w,\ell},$$

so $s_{i,\ell}' - s_{i,\ell} < v$. By a similar reasoning, $s_{i,\ell} - s_{i,\ell}' \leq c_{\ell,z}' - c_{\ell,z}$, and so $s_{i,\ell} - s_{i,\ell}' < v$. Putting everything together, $|s_{i,\ell}' - s_{i,\ell}| < v$.

Case 4: Action a_ℓ is played in s_i but it is not in s_i'. The argument is similar to case 3. $\square$

B.3. **Proof of Theorem 1.** The theorem relies on the concept of a q-contraction.

Definition 9. *Let (X, d) be a metric space. If $\xi : X \to X$ satisfies the condition $d(\xi(x), \xi(y)) < d(x, y)$ for all $x, y \in X$ such that $d(x, y) > q$, then ξ is called a q-contraction.*

A traditional contraction mapping "shrinks" the distance between the images of all points. A q-contraction only "shrinks" the distance between the images of points that are sufficiently far apart (further apart than q). Naturally, a q-contraction cannot have fixed points that are too far apart.

Proof of Theorem 1. Recall that i's expected utility of playing a_i when his type is t_i and the other players play s_{-i} is given by (B.1). Now pick $n, m \in \{1, \ldots, M_i\}$ such that $n > m$. If it exists, the fictitious cutoff between $a_{i,n}$ and $a_{i,m}$ is defined as the type t_i such that

$$Eu_i(a_{i,m}, s_{-i}, t_i) = Eu_i(a_{i,n}, s_{-i}, t_i),$$

that is,

$$\int_{\mathbb{R}} \sum_{\gamma \geq \underline{\gamma}} \Delta_m^n u_i(\gamma, \theta) g_i(\gamma | \theta, s_{-i}, c_{n,m}) d\mu_i(\theta | c_{n,m}) = 0. \tag{B.14}$$

By state monotonicity, $\Delta_m^n u_i$ is strictly increasing in θ and increasing in γ. Since μ_i is strictly increasing in t_i w.r.t. first-order stochastic dominance, and since G_i is increasing in (θ, t_i)

w.r.t. to first-order stochastic dominance, there can be only one type t_i that satisfies (B.14). As a result, the best-replies (which are cutoff strategies) are almost everywhere functions, and not correspondences. Consider two profiles of strategies for players $-i$, $s_{-i} = (s_{j,\ell})$ and $s'_{-i} = (s'_{j,\ell})$. Denote $v_{j,\ell} = |s'_{j,\ell} - s_{j,\ell}|$ for $\ell = 1, \ldots, M_j - 1$. Let $v = \max_{j \neq i} \max_{\ell \in \{1, \ldots, M_j - 1\}} v_{j,\ell}$. Player i's cutoff between $a_{i,n}$ and $a_{i,m}$ against s_{-i}, denoted $c_{n,m}$, satisfies (B.14). The cutoff between $a_{i,n}$ and $a_{i,m}$ against s'_{-i} is $c'_{n,m}$. By way of contradiction, assume $c'_{n,m} = c_{n,m} + v$. Hence

$$\int_{\mathbb{R}} \sum_{\gamma \geq \underline{\gamma}} \Delta^n_m u_i(\gamma, \theta) g_i(\gamma | \theta, s'_{-i}, c_{n,m} + v) d\mu_i(\theta | c_{n,m} + v) = 0. \tag{B.15}$$

If $v > \varepsilon(\mu, \boldsymbol{u})$, Proposition 2 says that (B.14) and (B.15) cannot hold simultaneously. That is, $c'_{n,m} = c_{n,m} + v$ cannot be the cutoff against s'_{-i} if $c_{n,m}$ is the cutoff against s_{-i}. Clearly, this claim holds for $c'_{n,m} \geq c_{n,m} + v$. Therefore, $c'_{n,m} - c_{n,m} < v$. If $c'_{n,m}$ is the cutoff against s'_{-i}, the same argument shows that whenever $c_{n,m}$ is larger than $c'_{n,m} + v$, both cannot cutoffs. In conclusion, if $v > \varepsilon(\mu, \boldsymbol{u})$, then for all players, $|c'_{n,m} - c_{n,m}| < v$ for all n, m such that both cutoffs exist. Proposition 4 implies that each i's best-reply is an $\varepsilon(\mu, \boldsymbol{u})$-contraction. From Milgrom and Roberts (1990), it follows that there exist two extremal equilibria, $\overline{s}$ and $\underline{s}$, that correspond to the extremal profiles of rationalizable strategies. We abuse notation and use d as the sup-norm on different metric spaces. Let $\boldsymbol{e} = (1, 1, \ldots, 1)$ be the vector of ones. Since br_i is an $\varepsilon(\mu, \boldsymbol{u})$-contraction, if $d(\overline{s}, \underline{s}) > \varepsilon(\mu, \boldsymbol{u})$, then we have $d(br_i(\underline{s}_{-i} - d(\overline{s}, \underline{s})\boldsymbol{e}), br_i(\underline{s}_{-i})) < d(\overline{s}, \underline{s})$. Thus,

$$
\begin{aligned}
d(\overline{s}, \underline{s}) &= d(br(\overline{s}), br(\underline{s})) \\
&= \max_{i \in N} d(br_i(\overline{s}_{-i}), br_i(\underline{s}_{-i})) \\
&\leq \max_{i \in N} d(br_i(\underline{s}_{-i} - d(\overline{s}, \underline{s})\boldsymbol{e}), br_i(\underline{s}_{-i}))) \\
&< d(\overline{s}, \underline{s}),
\end{aligned}
$$

where the first inequality holds because best-replies are increasing.[23] This string of inequalities leads to a contradiction, and thus $d(\overline{s}, \underline{s}) \leq \varepsilon(\mu, \boldsymbol{u})$. $\qquad \square$

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[23]Notice $\underline{s}_{-i} - d(\overline{s}, \underline{s})$ is a larger strategy than $\overline{s}_{-i}$.

B.4. **Theorem 2.** We first state a proposition that will be used in the proof.

Proposition 5. *Let $\{c_{n,m}\}$ be the set of fictitious cutoffs under μ, and let $\{c_{n,m}\}$ be the set of fictitious cutoffs under μ', where μ'_i is more optimistic than μ_i for each i. If, for some $v > 0$, $c_{n,m} - c'_{n,m} \geq v$ for all n and m such that both fictitious cutoffs exist, then $s_{i,\ell} - s'_{i,\ell} \geq v$ for all $\ell = 1, \ldots, M_i - 1$.*

The proof is similar to that of Proposition 4.

Proof of Theorem 2. In supermodular games, the largest (smallest) equilibrium coincide with the largest (smallest) profile of rationalizable strategies. Consider the largest (smallest) equilibrium, denoted by $\overline{s}$ ($\underline{s}$), under beliefs $\{\mu_i\}$. Against $\overline{s}_{-i}$, i's fictitious cutoff between a_n and a_m satisfies

$$\int_{\mathbb{R}} \sum_\gamma \Delta^n_m u_i(\gamma, \theta) g_i(\gamma|\theta, \overline{s}_{-i}, c_{n,m}) d\mu_i(\theta|c_{n,m}) = 0. \tag{B.16}$$

Since belief μ'_i is more optimistic than μ_i,

$$\int_{\mathbb{R}} \sum_\gamma \Delta^n_m u_i(\gamma, \theta) g'_i(\gamma|\theta, \overline{s}_{-i}, c_{n,m}) d\mu'_i(\theta|c_{n,m}) d\theta \geq 0, \tag{B.17}$$

because $\Delta^n_m u_i$ is increasing in θ and γ. Thus, the fictitious cutoff between a_n and a_m must be smaller under μ'_i than μ_i. Consider any s_{-i} and t_i such that

$$\int_{\mathbb{R}} \sum_\gamma \Delta^n_m u_i(\gamma, \theta) g_i(\gamma|\theta, s_{-i}, t_i) d\mu_i(\theta|t_i) = 0. \tag{B.18}$$

If for $v \geq 0$, we have

$$\int_{\mathbb{R}} \sum_\gamma \Delta^n_m u_i(\gamma, \theta) g'_i(\gamma|\theta, \overline{s}_{-i}, t_i - v) d\mu'_i(\theta|t_i - v) d\theta > 0, \tag{B.19}$$

then $t_i - v$ cannot be the fictitious cutoff under μ'_i, because $t_i - v$ is too large. This means that v should be increased. Before determining how large v can be, we consider three strings of inequalities.

The first string of inequalities goes as follows: for every t_i, n and m, and v,

$$\int_{\mathbb{R}} \sum_\gamma (\Delta^n_m u_i(\gamma, \theta + \omega_1 - \psi^1_i(v)) - \Delta^n_m u_i(\gamma, \theta)) g'_i(\gamma|\theta + \omega_1 - \psi^1_i(v), \overline{s}_{-i}, t_i - v) d\mu_i(\theta|t_i)$$

$$= (a_n - a_m) \int_{\mathbb{R}} \sum_{\gamma} \left(\frac{\Delta_m^n u_i(\gamma, \theta + \omega_1 - \psi_i^1(v)) - \Delta_m^n u_i(\gamma, \theta)}{a_n - a_m} \right) \times$$

$$g_i'(\gamma | \theta + \omega_1 - \psi_i^1(v), \overline{s}_{-i}, t_i - v) d\mu_i(\theta | t_i)$$

$$\geq (a_n - a_m) \int_{\mathbb{R}} \sum_{\gamma} M_*(\omega_1 - \psi_i^1(v), t_i) g_i'(\gamma | \theta + \omega_1 - \psi_i^1(v), \overline{s}_{-i}, t_i - v) d\mu_i(\theta | t_i)$$

$$= (a_n - a_m) M_*(\omega_1 - \psi_i^1(v), t_i) \tag{B.20}$$

The second string of inequalities goes as follows. Let $G_i^*(\tau_i)$ be the cdf of $G_i(\tau_i) \vee G_i'(\tau_i - b(v))$ and $G_{*,i}(\tau_i)$ be the cdf of $G_i(\tau_i) \wedge G_i'(\tau_i - b(v))$. Use similar notation for the probability mass functions. Assuming that $G_i'(\tau_i - b(v)) \geq_{st} G_i(\tau_i)$ for all τ_i, we have for every τ_i, n and m, and v,

$$\sum_{\gamma \geq \underline{\gamma}} (G_i'(S(\gamma) | \tau_i - b(v)) - G_i(S(\gamma) | \tau_i)) \times$$

$$\left(\frac{\Delta_m^n u_i(\gamma, \theta) - \Delta_m^n u_i(S(\gamma), \theta)}{(a_n - a_m)(\gamma - S(\gamma))} \right) (a_n - a_m)(\gamma - S(\gamma))$$

$$\geq \sum_{\gamma \geq \underline{\gamma}} (G_i'(S(\gamma) | \tau_i - b(v)) - G_{*,i}(S(\gamma) | \tau_i)) \times$$

$$\left(\frac{\Delta_m^n u_i(\gamma, \theta) - \Delta_m^n u_i(S(\gamma), \theta)}{(a_n - a_m)(\gamma - S(\gamma))} \right) (a_n - a_m)(\gamma - S(\gamma))$$

$$\geq (a_n - a_m) \sum_{\gamma \geq \underline{\gamma}} (G_i'(S(\gamma) | \tau_i - b(v)) - G_{*,i}(S(\gamma) | \tau_i))(\gamma - S(\gamma)) C_*(\theta)$$

$$= (a_n - a_m) C_*(\theta) \sum_{\gamma} \gamma(g_i'(\gamma | \tau_i - b(v)) - g_{*,i}(\gamma | \tau_i))$$

$$\geq (a_n - a_m) C_*(\theta) w_i^2(v) \tag{B.21}$$

The third string of inequalities goes as follows. Assuming now $G_i'(\tau_i - b(v)) \not\geq_{st} G_i(\tau_i)$ for some τ_i, we have for all τ_i, n and m, and v,

$$\sum_{\gamma \geq \underline{\gamma}} (G_i'(S(\gamma) | \tau_i - b(v)) - G_i(S(\gamma) | \tau_i)) \times$$

$$\left(\frac{\Delta_m^n u_i(\gamma, \theta) - \Delta_m^n u_i(S(\gamma), \theta)}{(a_n - a_m)(\gamma - S(\gamma))} \right) (a_n - a_m)(\gamma - S(\gamma)) \tag{B.22}$$

$$\geq \sum_{\gamma \geq \underline{\gamma}} (G_i'(S(\gamma)|\tau_i - b(v)) - G_i^*(S(\gamma)|\tau_i)) \times$$

$$\left(\frac{\Delta_m^n u_i(\gamma, \theta) - \Delta_m^n u_i(S(\gamma), \theta)}{(a_n - a_m)(\gamma - S(\gamma))} \right) (a_n - a_m)(\gamma - S(\gamma))$$

$$\geq (a_n - a_m) \sum_{\gamma \geq \underline{\gamma}} (G_i'(S(\gamma)|\tau_i - b(v)) - G_i^*(S(\gamma)|\tau_i))(\gamma - S(\gamma))C_i^*(\theta)$$

$$= (a_n - a_m)C_i^*(\theta) \sum_{\gamma} \gamma(g_i'(\gamma|\tau_i - b(v)) - g_i^*(\gamma|\tau_i))$$

$$\geq (a_n - a_m)C_i^*(\theta)w_i^2(v). \tag{B.23}$$

Equations (B.21) and (B.23) imply

$$\int \sum_{\gamma \geq \underline{\gamma}} (G_i'(S(\gamma)|\tau_i - b(v)) - G_i(S(\gamma)|\tau_i))(\Delta_m^n u_i(\gamma, \theta) - \Delta_m^n u_i(S(\gamma), \theta))d\mu_i(\theta|t_i)$$

$$\geq (a_n - a_m) \min\{\omega_i^2(v)C_{i,*}(t_i), \omega_2(v)C_i^*(t_i)\} \tag{B.24}$$

Now we conclude. Suppose $v < \delta(\mu, \mu', \boldsymbol{u})$. Then, by definition of $\delta(\cdot)$,

$$(a_n - a_m)\big(E_{\theta|t_i}[M_{i,*}(\theta + \omega_i^1 - \psi_i^1(v))]$$

$$+ \min\{\omega_i^2(v)E_{\theta|t_i}[C_*(\theta)], \omega_2(v)E_{\theta|t_i}[C^*(\theta)]\}\big) > 0 \tag{B.25}$$

Since (B.18) holds, it follows from (B.25), (B.20) and (B.24) that

$$\int_{\mathbb{R}} \sum_{\gamma} (\Delta_m^n u_i(\gamma, \theta + \omega_1 - \psi_i^1(v)) - \Delta_m^n u_i(\gamma, \theta))g_i'(\gamma|\theta + \omega_1 - \psi_i^1(v), \overline{s}_{-i}, t_i - v)d\mu_i(\theta|t_i)$$

$$+ \int_{\mathbb{R}} \sum_{\gamma} \Delta_m^n u_i(\gamma, \theta)(g_i'(\gamma|\theta + \omega_1 - \psi_i^1(v), \overline{s}_{-i}, t_i - v) - g_i(\gamma|\theta, \overline{s}_{-i}, t_i))d\mu_i(\theta|t_i) > 0. \tag{B.26}$$

But (B.26) is equivalent to

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta_m^n u_i(\gamma, \theta + \omega_1 - \psi_i^1(v))g_i'(\gamma|\theta + \omega_1 - \psi_i^1(v), \overline{s}_{-i}, t_i - v)d\mu_i(\theta|t_i) > 0 \tag{B.27}$$

which is equivalent to

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta_m^n u_i(\gamma, \theta)g_i'(\gamma|\theta, \overline{s}_{-i}, t_i - v)d\mu_i(\theta - \omega_1 + \psi_i^1(v)|t_i) > 0. \tag{B.28}$$

This last equation implies (B.19). Therefore, the transition to beliefs μ' must lead each fictitious cutoff to increase by more than $v < \delta(\mu, \mu', \boldsymbol{u})$, for otherwise an increase would imply (B.19), a contradiction of optimality. □

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